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Critical point theory of symmetric functions and closed geodesics*

Mónica Clapp
Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D.F., México

Dieter Puppe[1]
Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Heidelberg, Germany

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Abstract: We develop a version of equivariant critical point theory particularly adapted to finding closed geodesics by variational methods and use it to improve the known lower bounds for the number of "short" closed geodesics on some closed Riemannian manifolds.

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0. Introduction

Let M be a Riemannian manifold. By a *parametrized closed geodesic* on M we mean a non-constant map γ of the unit circle $S^1 \subset \mathbb{C} = \mathbb{R}^2$ into M satisfying the usual differential equation for geodesics (in particular preserving arclength up to a constant factor). Its equivalence class under isometries of S^1 (elements of $O(2)$) is called a *closed geodesic*. Both γ and its equivalence class are called *prime* if γ cannot be written as a composition with $e^{it} \mapsto e^{int}$, $|n| \geqslant 2$.

One knows since 1951 that every closed Riemannian manifold has at least one closed geodesic (L. Lusternik and A.I. Fet [32, 20]). In many cases the topological structure of a manifold M guarantees the existence of an infinite number of prime closed geodesics for every Riemannian metric on M (we give a survey in Sect. 6). Recently this was proved for the 2-sphere $M = S^2$ (J. Franks [21] together with V. Bangert [10]).

On the other hand there are interesting manifolds M, including

$$\begin{array}{l} \text{the } m\text{-sphere } S^m \text{ with } m \geqslant 3, \\ \text{complex projective space } \mathbb{C}P^m \text{ with } m \geqslant 2, \\ \text{quaternionic projective space } \mathbb{H}P^m \text{ (with } m \geqslant 2) \\ \text{and the Cayley plane } \mathrm{Ca}\, P^2, \end{array} \tag{0.1}$$

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[1] *Corresponding author*: Dieter Puppe, Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany, *E-mail*: puppe@sun0.urz.uni-heidelberg.de.

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for which at present one can neither exclude the possibility that every Riemannian metric on M has infinitely many prime closed geodesics nor that there exists a Riemannian metric on M with exactly one prime closed geodesic.

What one knows about the manifolds listed in (0.1) are some lower bounds for the number of prime closed geodesics depending upon the deviation of the given metric from the standard one. These are the kind of results which we are going to improve here.

In order to give a first idea of our results let M be the standard m-sphere as a differentiable manifold, with some Riemannian metric (not necessarily the standard one). Consider all (ordinary) circles on $M = S^m$, i.e., non-empty intersections of $S^m \subset \mathbb{R}^{m+1}$ with a plane in $\mathbb{R}^{m+1}$. Let $l_{\max}$ be the maximal length of such a circle measured with respect to the (non-standard) metric on M. Then we show that M has at least $g(m)$ closed geodesics (not necessarily prime) of length $\leqslant l_{\max}$, where $g(m)$ is the integer defined by

$$g(m) := m + 2^q - 1, \quad 2^q \leqslant m < 2^{q+1}. \tag{0.2}$$

Such geodesics are sometimes called "short." Note that

$$\tfrac{1}{2}(3m - 1) \leqslant g(m) \leqslant 2m - 1. \tag{0.3}$$

So far we cannot rule out the possibility that all of these closed geodesics are just multiples of a single prime one. But then this prime closed geodesic would have length $\leqslant l_{\max}/g(m)$.

Reversing the argument, let $l_{\min}$ be the minimal length of a closed geodesic on M (which always exists, cf. 3.4 below). For any real number x we denote by $[x]$ the largest integer $\leqslant x$ and by $]x[$ the smallest $\geqslant x$. Then M has at least $]g(m)/p[$ prime closed geodesics, where $p = [l_{\max}/l_{\min}]$.

Compared to the fact that there are infinitely many prime closed geodesics if $m = 2$ and the possibility that the same holds for all $m > 2$, this is a weak result. Note however that one cannot expect infinitely many closed geodesics whose length is smaller than some given number. To see this, consider an ellipsoid in $\mathbb{R}^{m+1}$ defined by the equation

$$\sum_{i=1}^{m+1} \frac{x_i^2}{a_i^2} = 1$$

with the metric induced by the standard one on $\mathbb{R}^{m+1}$. Its $\frac{1}{2}m(m+1)$ principal ellipses are (prime) closed geodesics. There are always infinitely many other prime closed geodesics (cf., e.g., [31; 33.5.15]), but if the lengths of the half axes a_i are sufficiently close to each other and pairwise different then all of those have arbitrarily large length [38; Chap. IX, Theorem 4.1].

Already in 1953 S.I. Al'ber made a claim similar to our result explained above [2], [3; Sect. 24, Theorems 35 and 37] (cf. also [28; Theorem 1], [29; Theorem at the end of Sect. 3], [30; Theorem 2.3.6]), but the proofs turned out to be incorrect (cf. [6; Sect. 4], [8; Remark in Sect. 2] and Sect. 1 below). The mistakes are related to the possible occurrence of multiple (i.e., non-prime) closed geodesics with lengths between $l_{\min}$ and $l_{\max}$. This possibility is excluded if $l_{\max} < 2l_{\min}$, i.e., $p = 1$, and in this case a proof was given in [25]. There are other proofs in [6] together with [7] and in [8; Theorems A and B] under somewhat different hypotheses, but they also exclude the occurrence of multiple closed geodesics in the relevant range. This is not excluded in [8; Theorem C], but also there the hypotheses are different from ours and, at least in some cases, the lower bounds for the number of closed geodesics are smaller than ours. We will discuss this in more detail in Sect. 1, where we formulate all our results on closed geodesics. They include the case of projective spaces for which the improvement over what was known before is even better than for spheres.

Before doing this we explain the general method. It is a version of equivariant critical point theory which works well even if there are critical points with nontrivial isotropy group. In particular it allows to deal efficiently with multiple closed geodesics (which have non-trivial isotropy groups of the $O(2)$-action), but may also be interesting for other situations involving non-free group actions.

Let M be a closed (= compact without boundary) Riemannian $\mathcal{C}^\infty$-manifold. As usual we denote by ΛM the $\mathcal{C}^\infty$-Hilbert manifold of closed H^1-Sobolev curves $\gamma : S^1 \to M$ (cf. Sect. 3 for details). The energy function $E : \Lambda M \to \mathbb{R}$ is defined by

$$E(\gamma) := \tfrac{1}{2} \int_0^1 \left\| \frac{d}{dt} \gamma(e^{2\pi it}) \right\|^2 dt.$$

It is $\mathcal{C}^\infty$ and satisfies the Palais–Smale condition (cf., e.g., [31; 2.4.9] and Sect. 3 below).

A critical point γ of E with $E(\gamma) > 0$ is nothing else but a parametrized closed geodesic. The standard action of $O(2)$ on S^1 induces an action on ΛM leaving the energy function E invariant. Hence a closed geodesic (as defined in the beginning) is a critical $O(2)$-orbit of E with positive (critical) value.

The isotropy subgroup of $O(2)$ corresponding to some non-constant $\gamma \in \Lambda M$ intersects $SO(2)$ in a finite cyclic subgroup the order of which we call the *multiplicity* of γ. (Note that *multiplicity* in [3; Sect. 22, 2. Definition 9] means something completely different.) Closed curves with multiplicity 1 are called *prime*. The multiplicity of γ is divisible by n if and only if there is a closed curve $\tilde{\gamma}$ such that $\gamma(e^{it}) = \tilde{\gamma}(e^{int})$ for all $t \in \mathbb{R}$. Then γ will be called an *n-fold multiple of $\tilde{\gamma}$*.

The classical Lusternik–Schnirelmann method for finding lower bounds for the number of critical points of a function Φ on X consists in showing that a certain topological invariant, the *category* cat$X$, is such a lower bound and then giving further estimates for cat$X$. There are equivariant versions of this method which proved to be very useful for certain applications (cf. [15] and further references given there). In principle they may also be applied to the problem of closed geodesics. It seems impossible, however, to obtain our results by just looking at numerical invariants of $X := E^b := \{\gamma \in \Lambda M | E(\gamma) \leqslant b\}$ similar to cat$X$. What we will do is to relate the $O(2)$-equivariant topological structure of E^b to the critical orbits of E below b, carefully keeping track of the isotropy subgroups of $O(2)$ belonging to these orbits.

The essence of this has nothing to do with the particular group $O(2)$. So we formulate our general result in Sect. 2 for an arbitrary compact Lie group G acting on a suitable space with a suitable G-invariant function. In Sect. 3 we show that ΛM and E are suitable in this sense (for $G = O(2)$) and we specialize the results of Sect. 2 to this context. From this, in Sections 4 and 5, we derive the results on closed geodesics stated in Sect. 1 using the multiplicative structure of Borel cohomology of E^b. In Sect. 6, in order to complete the picture, we discuss the following questions: Which homotopy types are known to contain only manifolds with infinitely many prime closed geodesics? How "many" others are there and to which of these are our results applicable?

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1. Statement of the results on closed geodesics

Let M be a simply connected closed Riemannian $\mathcal{C}^\infty$-manifold and let $l_{\min}$ be the smallest length of a closed geodesic on M (which always exists, cf. 3.4 below). In the introduction we spoke of *circles* on the standard sphere $S^m \subset \mathbb{R}^{m+1}$ meaning non-empty intersections with planes in $\mathbb{R}^{m+1}$. In the future however, we will need them with a parametrization, and so we define C to be the (compact) subspace of ΛS^m whose elements are the one point curves and the (ordinary round) circles on S^m parametrized proportionally to arc length. The subspace of this whose elements are the *great circles* will be denoted by A. For any abelian group B let $H_*(\cdot\,; B)$ and $H^*(\cdot\,; B)$ be singular homology and cohomology resp. with coefficients in B. In particular we will consider $B = \mathbb{F}_2 = \mathbb{Z}/2$. The explanation behind the number $g(m)$ defined in (0.2) is that it is one more than the mod 2 cup-length of the orbit space of A under the action of $O(2)$, the "space of unparametrized great circles on S^m," which is canonically homeomorphic to the Grassmannian $G(2, m-1)$ of unoriented planes in $\mathbb{R}^{m+1}$ (cf. 4.1 below; the cup-length of a space X with respect to a ring R is the largest number of elements in the cohomology ring $H^*(X; R)$ which have positive dimension and a non-zero product). The following theorem will be proved in Sect. 4.

1.1. Theorem. *Let $m > 0$ be minimal with $H_m(M; \mathbb{F}_2) \neq 0$ and let $f : S^m \to M$ be a $\mathcal{C}^1$-map which is non-trivial in $H_m(\cdot\,; \mathbb{F}_2)$. Let $l_{\max}$ be the maximal length of the image of a circle under f. Then*

(1) *M has at least $g(m)$ closed geodesics with length $\leqslant l_{\max}$.*

(2) *Either for some $l \leqslant l_{\max}$ the space of prime closed geodesics on M of length l has positive dimension or there are at least $]g(m)/p[$ prime closed geodesics in M with pairwise different lengths $\leqslant l_{\max}$, where $p := [l_{\max}/l_{\min}]$.*

From either (1) or (2) one immediately obtains:

1.2. Corollary. *If M satisfies the hypotheses of Theorem 1.1, then it has at least $]g(m)/p[$ prime closed geodesics with length $\leqslant l_{\max}$.*

As we mentioned in the introduction, results similar to 1.1 were claimed by S.I. Al'ber in [2], [3; Sect. 24, Theorems 35 and 37] and by W. Klingenberg in [28; Theorem 1], [29; Theorem at the end of Sect. 3], [30; Theorem 2.3.6]). In [3; Sect. 24, Theorems 35 and 37 Parts 1), 2)] it was even asserted that the $g(m)$ closed geodesics found are "geometrically distinct" and in [28; Theorem 1] that they are "simple" which usually means "without self-intersection" but in this case probably meant the same as "prime" (compare [5; Sect. 9 Definition 25]). For these additional assertions no justification was given whatsoever, nor is there any known today. Later Klingenberg, in [29; Theorem at the end of Sect. 3] and [30; Theorem 2.3.6], changed the statement to claim only "arithmetically different" closed geodesics meaning just different closed geodesics in our sense (not necessarily prime). From the context in [3] one can conclude that Al'ber did not really mean what one usually calls "geometrically distinct" (i.e., having different images) and in [5; Sect. 9 Definition 24] he said so explicitely. Moreover, in [4; Theorem 6] and [5; Sect. 9 Theorem 39] he made much stronger hypotheses than in [3]. They obviously imply the hypotheses of our Theorem 1.1 and also $l_{\max} < 2l_{\min}$, i.e., $p = 1$. In addition they contain a condition on the sectional curvature of M. Today one knows that this is unnecessary

(N. Hingston [25] and our Theorem 1.1). (On the other hand [8; Theorem A] shows that one may drop the other hypotheses if M is homeomorphic to the m-sphere and the sectional curvature is "$\frac{1}{4}$-pinched." In [4; Theorem 6] and [5; Sect. 9, Theorem 39] the conditions on the sectional curvature are weaker than that, but D.V. Anosov points out that "this was due to an error in a reference on differential geometry used there" [7; Sect. 4, footnote 4, p. 435 of the English translation].)

But even in their most restricted form the claims were not really proved in the papers cited nor anywhere else before 1979. This was apparently first noticed by W. Ballmann in 1979 [9; p. 48 after (3.3)] and it is explained in [6; Sect. 4] and in [8; Remark in Sect. 2]. The problem is that Theorem 27 in [3; Sect. 18], the "Lemma of Al'ber," is definitely false. It claims that the inclusion of $A/O(2) \cong G(2, m-1)$ into the orbit space of the space $\Lambda S^m - S^m$ of non-constant closed curves on S^m induces a monomorphism in homology and, equivalently, an epimorphism in cohomology mod 2, and this is not true. Two ways out of this were found: One is to replace $\Lambda S^m - S^m$ by smaller subspaces; this was done by D.V. Anosov [6], [7] and by W. Ballmann, G. Thorbergson and W. Ziller [8]. The other is to replace the orbit spaces under the action of $O(2)$ by the corresponding Borel construction ("homotopy orbit space"), which was done by N. Hingston [25] and will be used here (Sect. 4).

But then, in both cases, another difficulty appears if one does not exclude multiple closed geodesics (cf. [8; Lemma 1.5.(i) compared to Proposition 2.5.(ii)] and [25; end of 2.4]). In the present paper it is reflected by the fact that in Proposition 4.3(1) one has to assume the "multiplicity" n to be odd.

The easiest way to deal with this is to work in a range where there are no multiple closed geodesics. In our setting this means $p = 1$. Then our assertions 1.1(1) and 1.2 are equivalent, and they are almost precisely the same as the "sphere-part" of [25; Theorem in Sect. 5], except that there the map $f : S^m \to M$ is required to be a *homotopy equivalence* (not only a monomorphism in mod 2 homology in dimensions $\leqslant m$).

Similar results are [8; Theorems A and B], where the absence of multiple closed geodesics is enforced by other hypotheses. In [8; Theorem A] this is the "1/4-pinching condition" for sectional curvature under which the result was also proved in [6] together with [7].

In [8; Theorem C] multiple closed geodesics are allowed. The manifold M is assumed to be (closed, simply connected and) a mod 2 homology n-sphere. Then there is a map $f : S^m \to M$ as in our Theorem 1.1 with $m = n$ (cf. Sect. 6). Next the injectivity radius $i(M)$ is supposed to be $\geqslant \pi$, so that certainly $l_{\min} \geqslant 2\pi$. Finally a "$1/p^2$-pinching condition" is assumed. There is no simple precise relation between this and our quotient $l_{\max}/l_{\min}$. Roughly speaking however, our $p := [l_{\max}/l_{\min}]$ corresponds to $p-1$ in [8; Theorem C]. Thus in a way, where we have the lower bound $]g(m)/p[$ they have $](m-1)/p[$. Though this comparison is doubtful for large p, it does make good sense for $p = 2$ because on both sides this is the "first" case where multiple closed geodesics may appear. We give a small table to show how in this case our lower bounds compare to the ones in [8; Theorem C]:

m	2	3	4	5	6	7	8	$\cdots$	15	16	$\cdots$	31	32
$g(m)$	3	4	7	8	9	10	15		22	31		46	63
$]g(m)/2[$	2	2	4	4	5	5	8		11	16		23	32
$](m-1)/2[$	1	1	2	3	3	4	4		7	8		15	16

Note also that our Theorem 1.1 is very close to Al'ber's original claim [3; Sect. 24, Theorems 35 and 37 Parts 1), 2)] provided one uses Al'ber's own though unusual interpretation of the term "geometrically distinct," closer in any case than all the other results which were proved until now.

Now we turn to the case of projective spaces. Let $\mathbb{F}$ be either the complex numbers, the quaternions or the Cayley numbers and $d = 2, 4$ or 8 be the real dimension of $\mathbb{F}$. On the corresponding projective space $\mathbb{F}P^m$ (where $m \leqslant 2$ if $d = 8$) there is a canonical Riemannian metric such that every projective line in $\mathbb{F}P^m$ is isometric to the standard d-sphere (cf., e.g., [11; Chap. 3]). By a circle on $\mathbb{F}P^m$ we mean a circle on some projective line in $\mathbb{F}P^m$ in the same sense as before. If $2^q \leqslant m < 2^{q+1}$ define

$$g_2(m) := 2m + 2 \cdot 2^q - 1,$$

$$g_4(m) := m + 6 \cdot 2^q - 3 + \max\{m, 2^q + 2\},$$

$$g_8(1) := 15\ (= g(8)),$$

$$g_8(2) := 31.$$

In Section 5 we shall prove the following theorem (for M and $l_{\min}$ as at the beginning of this section):

1.3. Theorem. *Let $f : \mathbb{F}P^m \to M$ be a $\mathcal{C}^1$-map which induces an isomorphism in $H_*(\,\cdot\,; \mathbb{F}_2)$ and let $l_{\max}$ be the maximal length of the image of a circle under f. Then*

(1) M has at least $g_d(m)$ closed geodesics with length $\leqslant l_{\max}$.

(2) Either for some $l \leqslant l_{\max}$ the space of prime closed geodesics on M of length l has positive dimension or there are at least $]g_d(m)/p[$ prime closed geodesics in M with pairwise different lengths $\leqslant l_{\max}$, where $p := [l_{\max}/l_{\min}]$.

From either (1) or (2) one immediately obtains:

1.4. Corollary. *If M satisfies the hypotheses of Theorem 1.3, then it has at least $]g_d(m)/p[$ prime closed geodesics with length $\leqslant l_{\max}$.*

In 1965 W. Klingenberg [28; Theorem 2] and [29; Theorem at the end of Sect. 4] claimed a result similar to 1.3 for numbers which are slightly smaller in most cases and equal in a few others to $g_d(m)$, namely

$$g_d^K(m) := m + (2d - 1) \cdot 2^q - 1 \tag{1.5}$$

(which is one more than the mod 2 cup-length of the space of *unparametrized* great circles on $\mathbb{F}P^m$, cf. 5.5 below). Like in the case of spheres it was asserted in [28; Theorem 2] that the $g_d^K(m)$ closed geodesics found are "simple." In [29] this was changed into "arithmetically different." But even in this form the proofs were incorrect for the same reasons as in the case of spheres.

For $p = 1$, the assertions 1.3(1) and 1.4 are equivalent. In this case they are essentially contained in [25; Theorem in Sect. 5] for Klingenberg's numbers instead of $g_d(m)$. Again, as in the sphere case above, Hingston requires the map $f : \mathbb{F}P^m \to M$ to be a *homotopy equivalence* (not only an isomorphism in homology mod 2). Also, Hingston's proof contains a certain gap (cf. Remark 5.10) which we fill in 5.6.

Still for $p = 1$, H.-B. Rademacher [42] proved that 1.4 holds if Klingenberg's numbers are replaced by

$$g_d^R(m) := 2m + (2d-2) \cdot 2^q - d + 1 \tag{1.6}$$

(which is one more than the mod 2 cup-length of the space of *unoriented* parametrized great circles on $\mathbb{F}P^m$, cf. 5.3 below) and M has trivial homology in dimensions $< d$ with *integer* coefficients (not only with mod 2 coefficients). Our number $g_d(m)$ is the maximum of $g_d^K(m)$ and $g_d^R(m)$.

For $p > 1$ our results are new. We give a table for $p = 2$:

m	1	2	3	4	5	6	7	8	$\cdots$	14	15	16
$]g_2(m)/2[$	2	4	5	8	9	10	11	16		22	23	32
$]g_4(m)/2[$	4	8	8	16	16	17	18	32		37	38	64
$]g_8(m)/2[$	8	16										

It remains to compare this with some results in [8]. We need not look at $m = 1$ since this is just the sphere case in which 1.3 is contained in 1.1 and which was thoroughly discussed after 1.1 and 1.2. For $m \geqslant 2$ the relevant result in [8] is Theorem 3.3, part of which is Theorem D. The lower bounds for the number of (prime) closed geodesics on a projective space $\mathbb{F}P^m$ which one obtains from there do not vary with m as soon as $m \geqslant 3$. If multiple closed geodesics are excluded [8; Theorem 3.3(ii)] these bounds are much smaller than those of either [25] or [42] and hence of 1.3, 1.4 above. If multiple closed geodesics do occur [8; Theorem 3.3(i)], the bounds are independent of m for all $m \geqslant 2$. In this case there is no further information on $\mathbb{C}P^m$ (except the existence of one closed geodesic). For $\mathbb{H}P^m$ one gets two prime closed geodesics if multiplicities up to 3 are allowed and no information otherwise. For $\mathrm{Ca}P^2$ one gets $]8/p[$ prime closed geodesics if multiplicities up to p are allowed.

In contrast to this, for every fixed p and for $d = 2$ or 4, our numbers $]g_d(m)/p[$ in 1.3 and 1.4 are unbounded as functions of m.

1.7. Supplement to 1.3 and 1.4. In Theorem 1.3 and Corollary 1.4 the hypothesis that $f : \mathbb{F}P^m \to M$ induces an isomorphism in $H_*(\,\cdot\,; \mathbb{F}_2)$ partly was made for simplicity. In fact it may be replaced by a weaker one: Except in the case where $\mathbb{F} = \mathbb{H}$ ($d = 4$) and $m = 2^q + 1 \geqslant 2$ it is enough to have

$$\begin{aligned} &H^k(M; \mathbb{F}_2) = 0 \ \text{ for } \ 0 < k < d, \ \text{ and} \\ &f^* : H^d(M; \mathbb{F}_2) \longrightarrow H^d(\mathbb{F}P^m; \mathbb{F}_2) \cong \mathbb{F}_2 \ \text{ is non-zero.} \end{aligned} \tag{1.8}$$

Because of the multiplicative structure of $H^*(\mathbb{F}P^m)$ this implies that f^* is surjective in all dimensions, but it need not be injective.

In order to get the conclusions of 1.3 and 1.4 in the remaining case where $\mathbb{F} = \mathbb{H}$ and $m = 2^q + 1 \geqslant 2$ we have to assume in addition that

$$f^* : H^k(M; \mathbb{F}_2) \longrightarrow H^k(\mathbb{H}P^m; \mathbb{F}_2) \ \text{ is injective for } \ 4 \leqslant k \leqslant 8,$$

which together with (1.8) means that f induces an isomorphism in $H_*(\,\cdot\,; \mathbb{F}_2)$ for all $k \leqslant 8$ (compare Lemma 5.6). On the other hand, if in this particular case one replaces $g_4(m) = 2^{q+3}$

by Rademacher's number $g_4^R(m)$, which is one less, then (1.8) is again sufficient. This shows that Rademacher's result [42; 6.1 and 6.2] is completely contained in ours.

2. Critical point theory

First we recall some definitions and facts from [15] with minor changes. Let G be a compact Lie group and L a G-space. (In our present application G will be $O(2)$ and $L = \Lambda M$ the Hilbert manifold of closed H^1-Sobolev curves on a closed smooth manifold M.)

If X and Y are G-subspaces of L, we say that X is *G-deformable into Y (within L)* if there is a G-homotopy $\eta_t : X \to L$, $0 \leqslant t \leqslant 1$, such that $\eta_0(x) = x$, for all $x \in X$, and $\eta_1(X) \subset Y$.

If L' is another G-subspace of L and $\eta_t(x) \in L'$ for all $x \in X \cap L'$ and all t we say that the deformation *preserves L'*.

Let now $\Phi : L \to \mathbb{R}$ be a G-invariant function and K be a G-subset of L. For any $c \in \mathbb{R}$ we use the notation

$$\Phi^c := \{x \in L \mid \Phi(x) \leqslant c\},$$

$$K_c := \{x \in K \mid \Phi(x) = c\}.$$

2.1. Definition. We say that $\Phi : L \to \mathbb{R}$ has the *G-deformation property with respect to K between a and b* if the following conditions hold:

(D_0) There is an $\varepsilon > 0$ such that $\Phi^{a+\varepsilon}$ is G-deformable into Φ^a preserving Φ^a.

(D_1) The set K_c is compact for all $c \in \,]a, b]$.

(D_2) For every $c \in \,]a, b]$ and every G-neighborhood V of K_c there is an $\varepsilon > 0$ such that $\Phi^{c+\varepsilon} - V$ is G-deformable into $\Phi^{c-\varepsilon}$ preserving Φ^a.

This property is satified for all $b > a$ if, roughly speaking, L is a paracompact complete G-Banach manifold, Φ is a $\mathcal{C}^1$-function satisfying the Palais–Smale condition, K is the critical set of Φ, and a is a regular value of Φ. A precise formulation and a proof are given in [15; Appendix A]. We shall not use this general result but verify in Sect. 3 more directly that the G-deformation property is satisfied in our present application.

Recall [40; Sect. 6] that a G-ANR is a metrizable G-space X such that every G-map $Z \to X$ from a closed G-subspace Z of a metrizable G-space Y into X has a G-extension $U \to X$ to a G-neighborhood U of Z in Y. This condition is satisfied in many interesting situations, cf. [40] or [15; Appendix B] for details. In particular it will be satisfied in our present application (cf. 3.2).

Now we formulate and prove the basic theorem which replaces and refines the usual Lusternik–Schnirelmann method for our present purposes.

2.2. Theorem. *Assume that L is a G-ANR and that $\Phi : L \to \mathbb{R}$ satisfies the G-deformation property with respect to K between a and b. Then one of the following two conditions holds:*

I(a, b): *There is a number $c \in \,]a, b]$ such that the orbit space K_c/G of K_c has positive dimension.*

II(a, b): *For some $e > b$ and some $c_1 < c_2 < \cdots < c_k$ in the interval $]a, b]$ there are G-invariant open subsets $V_0, V_1, \ldots, V_k$ of L such that*

(i) $\Phi^e \subset V_0 \cup V_1 \cup \cdots \cup V_k$,

(ii) $\Phi^a \subset V_0$ *and* V_0 *is* G*-deformable (within* L*) into* Φ^a *preserving* Φ^a,

(iii) *for each* $i = 1, \ldots, k$ *there is a finite set* $F_i \subset K_{c_i}$ *such that* V_i *is* G*-deformable (within* L*) into*

$$G \cdot F_i \cong \coprod_{x \in F_i} G/G_x$$

where G_x *is the isotropy subgroup of* x.

Proof. Assume that neither I(a, b) nor II(a, b) holds. Let d be the greatest lower bound of those $c \in]a, b]$ for which II(a, c) (i.e., the second condition with b replaced by c) does not hold. Then $d > a$, because with $\varepsilon > 0$ as in 2.1(D_0) every $c \in]a, a+\varepsilon[$ satisfies II(a, c) with $e = a + \varepsilon$, $k = 0$ and $V_0 = \Phi^{a+\varepsilon}$. For any $x \in K_d$ the orbit $Gx \cong G/G_x$ is a G-ANR [40; 6.10 and 7.2], hence it has a G-neighborhood U_x in L with a G-retraction $r : U_x \to Gx$. Since L is also a G-ANR by hypothesis, there is a (possibly) smaller G-neighborhood U'_x of Gx such that the composition

$$U'_x \hookrightarrow U_x \xrightarrow{r} Gx \hookrightarrow L$$

is G-homotopic to the inclusion, i.e., U'_x is G-deformable into Gx. Since K_d is compact (by (D_1) in 2.1) there is a finite set $F \subset K_d$ such that $(U'_x | x \in F)$ is a covering of K_d. Since K_d/G has dimension 0, there is a *disjoint* family $(W_x | x \in F)$ of open G-sets such that $W_x \subset U'_x$ and still

$$K_d \subset \bigcup_{x \in F} W_x =: W.$$

Then W is G-deformable into $G \cdot F$. Choose a closed G-neighborhood $V \subset W$ of K_d. By (D_2) in 2.1 there is an $\varepsilon \in]0, d-a[$ and a G-deformation $\eta_t : \Phi^{d+\varepsilon} - V \longrightarrow L$, $0 \leqslant t \leqslant 1$, into $\Phi^{d-\varepsilon}$ preserving Φ^a. Let $V_0, V_1, \ldots, V_k$ be as in II$(a, d-\varepsilon)$ (which holds by the definition of d). Then $\eta_1^{-1}(V_0), \eta_1^{-1}(V_1), \ldots, \eta_1^{-1}(V_k)$, W is a covering of $\Phi^{d+\varepsilon}$ which shows that II(a, c) holds for all $c \in [d, d+\varepsilon[$. This contradicts the definition of d and finishes the proof. $\square$

Note that if one of the numbers c_i in II(a, b) is not a "critical value," i.e., not an element of $\Phi(K)$, then K_{c_i} and hence V_i is empty. So the content of the theorem does not change if we require c_i to be a critical value for all i. Therefore it guarantees the existence of either infinitely many critical orbits or of more than r critical values if one can show that II(a, b) is compatible with the topological structure of the G-pair (Φ^b, Φ^a) only if $k > r$. One way to do this uses the same idea as the cup-length estimate for Lusternik–Schnirelmann category and works as follows.

2.3. Let h^* be a multiplicative cohomology theory on the category of G-spaces as in [15; 4.1]. In our applications h^* will be Borel cohomology with $\mathbb{F}_2$-coefficients which is defined as

$$H^*_G(X; \mathbb{F}_2) := H^*(EG \underset{G}{\times} X; \mathbb{F}_2)$$

= singular cohomology with $\mathbb{F}_2$-coefficients of the orbit space $EG \underset{G}{\times} X$ of $EG \times X$ (with the diagonal action). Here EG is the total space of the universal principal G-bundle $EG \longrightarrow BG$. See [16; III.1], [25; Sect. 1] or [15; 5.4] for more details.

2.4. Proposition. *Assume that* $\mathrm{II}(a, b)$ *of* 2.2 *holds and that for each* $i \in \{1, \ldots, k\}$ *we have a cohomology class* $\alpha_i \in h^*(L)$ *which restricts to* 0 *on* $Gx \subset L$ *for each* $x \in F_i$. *Then*

$$h^*(L, \Phi^a) \cdot \alpha_1 \cdots \alpha_k$$

restricts to 0 *on* (Φ^e, Φ^a).

Proof. Since $G \cdot F_i$ is the disjoint union of all Gx for $x \in F_i$, the class α_i restricts to 0 on $G \cdot F_i$. Since V_i is G-deformable into $G \cdot F_i$ the same class also restricts to 0 on V_i. From the cohomology exact sequence of the pair (L, V_i) one gets a preimage $\tilde{\alpha}_i \in h^*(L, V_i)$ of α_i.

Any $\sigma \in h^*(L, \Phi^a)$ restricts to 0 on (V_0, Φ^a) because there is a G-deformation of V_0 into Φ^a preserving Φ^a. The cohomology exact sequence of the triple (L, V_0, Φ^a) gives a preimage $\tilde{\sigma} \in h^*(L, V_0)$ of σ.

The restriction of the product $\sigma\alpha_1 \ldots \alpha_k \in h^*(L, \Phi^a)$ to $h^*(\Phi^e, \Phi^a)$ is the same as that of $\tilde{\sigma}\tilde{\alpha}_1 \cdots \tilde{\alpha}_k \in h^*(L, V_0 \cup V_1 \cup \cdots \cup V_k)$ hence 0 since $\Phi^e \subset V_0 \cup V_1 \cup \cdots \cup V_k$. □

In the applications of Proposition 2.4 one starts, of course, the other way around. Usually one knows that a certain product of cohomology classes does not vanish and one wants to guarantee a certain number of critical values or at least orbits. If it is not known in advance which isotropy groups occur in the critical set K the condition in 2.4 that α_i restricts to 0 on each orbit in $F_i \cdot G$ may be hard to handle. We give a general description which looks a little complicated. Fortunately, it will simplify considerably in our specific applications.

For any G-set X let $\mathcal{I}(X)$ be the set of those subgroups H of G which occur as isotropy groups in X. For any $\beta \in h^*(L)$ let $\mathcal{N}(\beta)$ be the set of all subgroups H of G with the property that β restricts to 0 on every G-orbit of L with isotropy group H.

Suppose now that $\beta_1, \ldots, \beta_r \in h^*(L)$ are such that, for every $e > b$, $h^*(L, \Phi^a) \cdot \beta_1 \cdots \beta_r$ does not restrict to 0 on (Φ^e, Φ^a). Then, if condition $\mathrm{II}(a, b)$ of 2.2 holds there is no injective map $\mu : \{1, \ldots, k\} \longrightarrow \{1, \ldots, r\}$ with $\mathcal{I}(K_{c_i}) \subset \mathcal{N}(\beta_{\mu(i)})$ for all $i \in \{1, \ldots, k\}$. A well known theorem in combinatorics says

2.5. Proposition. *An injective map as above exists if and only if all subsets* I *of* $\{1, \ldots, k\}$ *satisfy*

$$\operatorname{card} I \leqslant \operatorname{card}\{j \in \{1, \ldots, r\} \mid \mathcal{I}(K_{c_i}) \subset \mathcal{N}(\beta_j) \textit{ for at least one } i \in I\}.$$

This is related to the so-called marriage problem. For a proof cf., e.g., [12; Sect. 2].

Now, Theorem 2.2 and Propositions 2.4 and 2.5 together give

2.6. Theorem. *Assume that* L *is a* G-*ANR and that* $\Phi : L \to \mathbb{R}$ *satisfies the* G-*deformation property with respect to* K *between* a *and* b. *Furthermore let* h^* *be a multiplicative cohomology theory on the category of* G-*spaces and suppose that there exist classes* $\beta_1, \ldots, \beta_r \in h^*(L)$ *such that, for every* $e > b$, $h^*(L, \Phi^a) \cdot \beta_1 \cdots \beta_r$ *does not restrict to* 0 *on* (Φ^e, Φ^a). *Then one of the following two conditions holds*:

(a) *There is a critical value* $c \in \,]a, b]$ *such that the orbit space* K_c/G *of* K_c *has positive dimension.*

(b) *There are critical values* $c_1 < c_2 < \cdots < c_k$ *in* $]a, b]$ *such that*

$$k > \operatorname{card}\{j \in \{1, \ldots, r\} \mid \mathcal{I}(K_{c_i}) \subset \mathcal{N}(\beta_j) \textit{ for at least one } i \in \{1, \ldots, k\}\}.$$

How can one use this? We discuss some special cases where we always assume the hypotheses of Theorem 2.6 to be true and indicate the additional assumptions.

2.7. Suppose that the action of G on $K \cap \Phi^{-1}]a, b]$ is free. Then $\mathcal{I}(K_{c_i})$ contains only the unit group and the condition $\mathcal{I}(K_{c_i}) \subset \mathcal{N}(\beta_j)$ just means that β_j restricts to 0 under each G-embedding $G \to K_{c_i}$. In Borel cohomology $h^*(X) = H^*(EG \underset{G}{\times} X)$ we have $h^*(G) = H^*(EG) = H^*(\mathrm{pt})$, so that the condition holds for all $\beta_j \in h^n(L)$ with $n > 0$ (where H^* is any cohomology theory satisfying the dimension axiom, e.g., singular cohomology with any coefficients).

2.8. It can happen that $\mathcal{N}(\beta_j) \supset \mathcal{I}(K \cap \Phi^{-1}]a, b])$ for all $j \in \{1, \dots, r\}$. Then 2.6(b) gives $k > r$. In particular this is true if β_j restricts to 0 on each orbit not reduced to a point, and the action of G on $K \cap \Phi^{-1}]a, b]$ has no fixed (= stationary) points.

2.9. In the proofs of the statements made in Sect. 1 we will only sometimes have the situation of 2.8. However, it will always be the case that for a certain pair $\mathcal{G}' \subset \mathcal{G}$ of sets of closed subgroups of G and for a certain number $r' \leqslant r$ we have

$$\mathcal{N}(\beta_j) \supset \begin{cases} \mathcal{G} & \text{for } j \leqslant r', \\ \mathcal{G}' & \text{for all } j \end{cases}$$

and $\mathcal{I}(K \cap \Phi^{-1}]a, b]) \subset \mathcal{G}$. Then condition (b) of 2.6 says that one of the following is true:

(b′) *There are more than r critical values in* $]a, b]$.

(b″) *There are more than r′ critical values c in* $]a, b]$ *with* $\mathcal{I}(K_c) \not\subset \mathcal{G}'$.

One obtains (b′) from 2.6(b) if $\mathcal{I}(K_{c_i}) \subset \mathcal{G}'$ for an least one i, and (b″) otherwise.

3. The case of closed geodesics

Let M be a paracompact finite-dimensional $\mathcal{C}^q$-manifold ($2 \leqslant q \leqslant \infty$) without boundary, and let ΛM be the set of continuous maps $\gamma : S^1 \to M$ such that for every $\mathcal{C}^q$-function $\varphi : M \to \mathbb{R}$ the composite $\varphi\gamma : S^1 \to \mathbb{R}$ is the integral of an L^2-function (i.e., it is absolutely continuous and its derivative is square integrable). It is well known (at least for $q = \infty$) that ΛM can be given the structure of a paracompact $\mathcal{C}^{q-1}$-Hilbert manifold (cf. [41; Sect. 13] or [31; 2.3 and 2.4]). An easy way to prove this which we have not seen in the literature is the following:

Take first $M = \mathbb{R}^n$ with the standard inner product $\langle\cdot, \cdot\rangle$ and the corresponding euclidean norm $\|\cdot\|$. Then $\Lambda\mathbb{R}^n$ is a (real) Hilbert space with the inner product

$$\langle\alpha, \beta\rangle_\Lambda := \int_0^1 \langle\alpha(e^{2\pi it}), \beta(e^{2\pi it})\rangle + \left\langle\frac{d}{dt}\alpha(e^{2\pi it}), \frac{d}{dt}\beta(e^{2\pi it})\right\rangle dt.$$

3.1. Lemma. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a $\mathcal{C}^q$-map ($1 \leqslant q \leqslant \infty$). Then ΛU is open in $\Lambda\mathbb{R}^n$ and $\gamma \mapsto f\gamma$ defines a $\mathcal{C}^{q-1}$-map $\Lambda f : \Lambda U \longrightarrow \Lambda\mathbb{R}^m$.*

Proof. It is an easy exercise to show $\|\gamma(z)\| \leqslant 2 \cdot \|\gamma\|_\Lambda$ for all $\gamma \in \Lambda\mathbb{R}^n$ and $z \in S^1$. This immediately implies that ΛU is open in $\Lambda\mathbb{R}^n$. Next one checks that $f\gamma \in \Lambda\mathbb{R}^m$ for all $\gamma \in \Lambda U$ (using $f \in \mathcal{C}^1$) and that Λf is continuous (using the above inequality and again $f \in \mathcal{C}^1$). Finally, if $q \geqslant 2$ and $f' : U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the derivative of f, we have

$$[(\Lambda f)'(\gamma) \cdot \xi](z) = [(\Lambda f')(\gamma)](z) \cdot \xi(z) = f'(\gamma(z)) \cdot \xi(z)$$

for all $\gamma \in \Lambda U$, $\xi \in \Lambda\mathbb{R}^n$ and $z \in S^1$. Then one proceeds by induction. $\square$

As in the beginning let now M be a paracompact finite-dimensional $\mathcal{C}^q$-manifold without boundary and $2 \leqslant q \leqslant \infty$. Choose a closed $\mathcal{C}^q$-embedding $M \subset \mathbb{R}^n$ for some n. There is an open neighborhood U of M in $\mathbb{R}^n$ and a $\mathcal{C}^q$-retraction $r : U \to M$ [39; 5.5 Theorem]. If $i : M \hookrightarrow U$ is the inclusion then $\Lambda(ir)$ is an idempotent $\mathcal{C}^{q-1}$-map of ΛU into itself, whose fixed point set is ΛM. Hence ΛM is a $\mathcal{C}^{q-1}$-Hilbert submanifold of $\Lambda U \subset \Lambda\mathbb{R}^m$, which is obviously paracompact.

If M' is another manifold like M with a closed $\mathcal{C}^q$-embedding $i' : M' \hookrightarrow \mathbb{R}^{n'}$ and $f : M \to M'$ a $\mathcal{C}^q$-map then $\Lambda(i'fr) : \Lambda U \longrightarrow \Lambda\mathbb{R}^{n'}$ is a $\mathcal{C}^{q-1}$-map, hence the same is true for its restriction $\Lambda f : \Lambda M \longrightarrow \Lambda M'$. In particular it follows that the $\mathcal{C}^{q-1}$-structure on ΛM does not depend on the choice of the embedding, of U, or of r (by taking $M' = M$ and $f = \mathrm{id}_M$), and that Λ is a functor.

The action of $O(2)$ on ΛM is given by

$$(g\gamma)(e^{it}) = \gamma(g^{-1}e^{it}), \quad g \in O(2), \gamma \in \Lambda M.$$

Its properties can be studied by again looking at a closed $\mathcal{C}^q$-embedding $M \subset \mathbb{R}^n$. The action $O(2) \times \Lambda M \longrightarrow \Lambda M$ is continuous, though not differentiable. But for every fixed $g \in O(2)$ the map $\gamma \mapsto g\gamma$ is a (bounded) linear operator on $\Lambda\mathbb{R}^n$, hence a $\mathcal{C}^{q-1}$-map on ΛM. Furthermore we have

3.2. Lemma. *ΛM is an $O(2)$-ANR.*

Proof. $\Lambda\mathbb{R}^n$ is an $O(2)$-AR [40; 6.5]. Hence the open subset ΛU (notation as above) and its retract ΛM are $O(2)$-ANRs [40; 6.7]; compare also [15; Appendix B]. $\square$

Now let a $\mathcal{C}^{q-1}$-Riemannian metric be given on M, so that we have the energy function $E : \Lambda M \longrightarrow \mathbb{R}$ defined by

$$E(\gamma) := \tfrac{1}{2} \int_0^1 \left\| \frac{d}{dt}\gamma(e^{2\pi it}) \right\|^2 dt,$$

where this time $\|\cdot\|$ is the norm belonging to the Riemannian metric on M. The energy is an $O(2)$-invariant $\mathcal{C}^{q-1}$-function and we may speak of its critical set

$$K := \{\gamma \in \Lambda M \mid (dE)(\gamma) = 0\}.$$

From now on we assume that M is compact and (for simplicity) that $q = \infty$.

3.3. Lemma. *The energy function E has the $O(2)$-deformation property (in the sense of 2.1) with respect to its critical set K between 0 and ∞ (i.e., between 0 and b for all $b > 0$). Furthermore $K \cap E^b$ is compact for all b.*

Proof. $E^0 \cong M$ is an ANR with trivial $O(2)$-action, hence an $O(2)$-ANR. In 3.2 we saw that ΛM is also an $O(2)$-ANR. Therefore E^0 has an $O(2)$-neighborhood V in ΛM which is $O(2)$-

deformable into E^0 leaving E^0 pointwise fixed. By the compactness of M there is an $\varepsilon > 0$ with $E^\varepsilon \subset V$. This proves (D_0) of 2.1.

It is well known [31; 2.4.9] that (since M is compact) the energy function satisfies the *Palais–Smale condition:*

$$\text{Any sequence } (\gamma_n) \text{ in } \Lambda M \text{ such that } (E(\gamma_n)) \text{ is bounded and } ((dE)(\gamma_n)) \text{ converges to zero has a convergent subsequence.} \tag{PS}$$

From this the compactness of $K \cap E^b$ for all b, hence in particular condition (D_1) of 2.1, follow immediately, and by well known methods one can also derive (D_2) from it [15; Appendix A]. □

Now we are in the position to apply the results of Sect. 2, in particular Theorem 2.6 to the case $L = \Lambda M$, $\Phi = E$, and $G = O(2)$. At the same time we want to pass from energy levels to length levels in ΛM. Deviating a little from standard notation we use $\Lambda^l M$ to denote the set of all $\gamma \in \Lambda M$ whose *length* (not energy) is $\leqslant l$. Length is a continuous function ℓ on ΛM defined by

$$\ell(\gamma) := \int_0^1 \left\| \frac{d}{dt}\gamma(e^{2\pi i t}) \right\| dt.$$

It is related to energy by the inequality $E(\gamma) \geqslant \frac{1}{2}[\ell(\gamma)]^2$ for all $\gamma \in \Lambda M$, and equality holds if ond only if the parametrization of γ is proportional to arc length; hence for such curves, in particular for geodesics, there is a bijective correspondence between energy and length levels.

Observe now that Theorem 2.2 guarantees the existence of at least one closed geodesic if for some $e \in]0, \infty[$ the space $E^e \subset \Lambda M$ is not $O(2)$-deformable into $E^0 = \Lambda^0 M = M$ preserving M. The latter is well known to be the case for every M. It is obvious if the fundamental group $\pi_1(M)$ does not vanish. For a simply connected manifold M there are several possibilities to argue. The most elementary is as follows: Since M is closed it is not contractible. So there is an $m > 1$ such that the homotopy group $\pi_m(M)$ does not vanish. If we take m minimal with this property then $\pi_{m-1}(\Lambda M) \neq 0 = \pi_{m-1}(M)$, which easily implies that there is an e such that E^e is not deformable into $E^0 = M$ even if the $O(2)$-action is neglected. Another way is to find a cohomology theory h^* as in 2.3 such that

$$h^*(\Lambda M, M) \longrightarrow h^*(E^e, M)$$

induced by the inclusion does not vanish. Actually this is always the case for singular cohomology with any (non-zero) coefficients and e sufficiently large. Then one can also apply Theorem 2.6 instead of 2.2.

Let $l_{\min}$ be the greatest lower bound of the lengths of closed geodesics. The Palais–Smale condition for the energy function implies the existence of a sequence (γ_n) of parametrized closed geodesics converging to an element $\gamma \in \Lambda M$ with length $l_{\min}$. This γ is either a (non-constant) parametrized closed geodesic or a single point $x \in M$. But the latter is excluded by well known local properties of Riemannian manifolds. So we have

3.4. Proposition. *Every closed Riemannian manifold has a closed geodesic with minimal length.*

For closed geodesics only the cyclic subgroups $R_n \cong \mathbb{Z}/n$ of $SO(2)$ occur as isotropy groups, i.e., in the notation of Sect. 2 we have $\mathfrak{I}(K - \Lambda^0 M) \subset \{R_n \mid n = 1, 2, \dots\}$. Theorem 2.6 here takes the form

3.5. Theorem. *Let M be a compact Riemannian $\mathcal{C}^\infty$-manifold and let h^* be a multiplicative cohomology theory on the category of $O(2)$-spaces. Suppose that for a certain number $l_{\max} \in]0, \infty[$ there exist classes $\beta_1, \ldots, \beta_r \in h^*(\Lambda M)$ such that $h^*(\Lambda M, M) \cdot \beta_1 \cdots \beta_r$ does not restrict to 0 on $(\Lambda^{l_{\max}} M, M)$. Then one of the following two conditions holds:*

(a) *There is an $l \in]0, l_{\max}]$ such that the space of prime closed geodesics on M of length l has positive dimension.*

(b) *There are k closed geodesics on M with lengths $l_1 < l_2 < \cdots < l_k \in]0, l_{\max}]$ and k is greater than the number of those $j \in \{1, \ldots, r\}$ for which there exists an $i \in \{1, \ldots, k\}$ such that $R_n \in \mathcal{N}(\beta_j)$ for every n which occurs as the multiplicity of a closed geodesics of length l_i.*

This admittedly somewhat complicated statement has the following simple corollaries and these are what we shall actually use in the applications:

3.6. Corollary. *If in addition to the hypotheses of Theorem 3.5 we have*

$$R_n \in \mathcal{N}(\beta_j) \quad \text{if } n \text{ is odd or } j \leqslant r_2$$

for all $n = 1, 2, \ldots$ and some fixed $r_2 \leqslant r$, then one of the following three conditions holds:

(a) *There is an $l \in]0, l_{\max}]$ such that the space of prime closed geodesics on M of length l has positive dimension.*

(b) *There are more than r different lengths $\leqslant l_{\max}$ of closed geodesics on M.*

(b_2) *There are more than r_2 different lengths $\leqslant \frac{1}{2} l_{\max}$ of closed geodesics on M.*

3.7. Corollary. *If furthermore, in addition to the hypotheses of Corollary 3.6, we have $r \leqslant 2r_2 + 1$ then*

(1) *M has at least $r + 1$ closed geodesics with length $\leqslant l_{\max}$.*

(2) *Either for some $l \leqslant l_{\max}$ the space of prime closed geodesics on M of length l has positive dimension, or there are at least $](r+1)/p[$ different lengths $\leqslant l_{\max}$ of prime closed geodesics on M, where $p := [l_{\max}/l_{\min}]$ and $l_{\min}$ is the minimal length of a closed geodesic.*

Proof of 3.7 from 3.6. If (a) or (b) in 3.6 holds then 3.7 is obvious. If (b_2) holds then there are at least $](r_2 + 1)/p_2[$ different lengths $\leqslant \frac{1}{2} l_{\max}$ of prime closed geodesics on M, where

$$p_2 := \left[\frac{l_{\max}}{2 l_{\min}}\right] \leqslant \frac{1}{2}\left[\frac{l_{\max}}{l_{\min}}\right] = \frac{p}{2}.$$

Assertion (2) in 3.7 follows because

$$\left]\frac{r_2+1}{p_2}\right[\geqslant \left]\frac{2(r_2+1)}{p}\right[\geqslant \left]\frac{r+1}{p}\right[.$$

Assertion (1) follows from the fact that every prime closed geodesic with length $\leqslant \frac{1}{2} l_{\max}$ has twice as many multiples with length $\leqslant l_{\max}$ as it has with length $\leqslant \frac{1}{2} l_{\max}$. Note that this argument does not give twice as many different *lengths* $\leqslant l_{\max}$ of closed geodesics as one had $\leqslant \frac{1}{2} l_{\max}$. □

Proof of 3.6 from 3.5. This is an instance of 2.9: One gets 3.6(b) from 3.5(b) if all the closed geodesics referred to in 3.5(b) have odd multiplicity. Otherwise one gets (b_2). □

Proof of 3.5. There are two obstructions which prevent 3.5 from being an immediate corollary of 2.6. One is the use of length instead of energy. The other is the word "prime" in 3.5(a). The first can be overcome by

3.8. Lemma. *Let M be a closed Riemannian C^∞-manifold and let $l > 0$, $e > \frac{1}{2}l^2$. Then $\Lambda^l M$ is $O(2)$-deformable into E^e (within itself and a fortiori within ΛM) preserving $\Lambda^0 M = M$.*

We prove this lemma below and first use it to proceed with the proof of 3.5.

This is done by applying Theorem 2.6 in the case $L = \Lambda M$, $G = O(2)$, $a = 0$ and $b = \frac{1}{2}l_{\max}^2$. The lemma shows (with $l = l_{\max}$) that the inclusion

$$(\Lambda^{l_{\max}} M, M) \hookrightarrow (\Lambda M, M)$$

is $O(2)$-homotopic to some map into (E^e, M) for any $e > b$. Therefore $h^*(\Lambda M, M)\beta_1 \cdots \beta_r$ does not restrict to 0 on (E^e, M), and the hypotheses of Theorem 2.6 are satisfied.

To justify the word "prime" in 3.5(a) let $K(l, n) \subset \Lambda M/O(2)$ be the set of closed geodesics on M with length l and multiplicity n. For given l there is an n_0 such that $K(l, n) = \emptyset$ for all $n > n_0$. If $K(l, 1), \ldots, K(l, n_0)$ all have dimension 0 the same follows for

$$K(l, n_0) \cup K(l, n_0 - 1) \cup \cdots \cup K(l, n_0 - q), \qquad q = 1, \ldots, n_0 - 1$$

by induction on q, because $K(l, n_0) \cup K(l, n_0 - 1) \cup \cdots \cup K(l, n_0 - q + 1)$ is closed in $K(l, n_0) \cup K(l, n_0 - 1) \cup \cdots \cup K(l, n_0 - q)$ (cf., e.g., [26; Chap. II,Sect. 3, Cor. 2]). Hence, if the space $K(l, n_0) \cup \cdots \cup K(l, 1)$ of all closed geodesics of length l has positive dimension then so has $K(l, n)$ for some n. Since $K(l, n)$ is homeomorphic to $K(l/n, 1)$ this shows: If 3.5(a) is false then so is 2.6(a) with $b = \frac{1}{2}l_{\max}^2$. Thus we have 2.6(b) which implies 3.5(b). □

Proof of Lemma 3.8. The result is contained in [6; Theorem 4 and Sect. 7]. There an $O(2)$-equivariant homotopy of ΛM into itself is constructed which starts with the identity, leaves the constant (one point) curves fixed, and moves each non-constant $\gamma \in \Lambda M$ along a family of reparametrizations of γ such that in the end the parameter is proportional to arc length. This means that 3.8 even remains valid if $e = \frac{1}{2}l^2$ is allowed.

For our version of 3.8 there is a much easier proof which we found after a discussion with Steffen Heinze and Benjamin Schweizer. We also use the idea of reparametrizing non-constant curves proportional to arc length, but it suffices to do this approximately. For the purpose of this proof we identify $\mathbb{R}/\mathbb{Z}$ with S^1 by $t \mapsto \exp(2\pi i t)$ and we represent elements of ΛM by periodic maps $\gamma : \mathbb{R} \to M$ with period 1. Choose $\varepsilon > 0$, and for every γ and every $s \in [0, 1]$ define the real functions f_γ and $F_{\gamma,s}$ by

$$f_\gamma(t) := \frac{\|\gamma'(t)\| + \varepsilon}{\ell(\gamma) + \varepsilon}, \qquad F_{\gamma,s}(t) := (1 - s)t + s\int_0^t f_\gamma(\tau)\,d\tau + s\int_0^1 \tau f_\gamma(\tau)\,d\tau.$$

Each function $F_{\gamma,s}$ is absolutely continuous with derivative $\geqslant \varepsilon/(\ell(\gamma) + \varepsilon)$ almost everywhere. One can check that $(\gamma, s) \longmapsto \gamma_s := \gamma \circ F_{\gamma,s}^{-1}$ is a continuous map from $\Lambda M \times [0, 1]$ to ΛM, and it is $O(2)$-equivariant due to the last term (integration constant) in the definition of $F_{\gamma,s}(t)$. Obviously γ_s has the same length for every s, and an easy calculation gives $E(\gamma_1) \leqslant \frac{1}{2}(\ell(\gamma) + \varepsilon)^2$. We obtain our lemma by choosing ε such that $\frac{1}{2}(l + \varepsilon)^2 < e$. □

4. Proof of Theorem 1.1 (Spheres)

If G is a topological group and h^* a multiplicative cohomology theory on the category of G-spaces (for a definition cf., e.g., [15; 4.1]) then for every G-space X the unique map $X \to \mathrm{pt}$ (= one point space) induces a canonical homomorphism of graded rings $h^*(\mathrm{pt}) \to h^*(X)$. Frequently we will use the same letter for an element of $h^*(\mathrm{pt})$ and its image under this homomorphism.

If H^* is a multiplicative cohomology theory on spaces without group action one can define a cohomology theory H_G^* on G-spaces by

$$H_G^*(X) = H^*(EG \underset{G}{\times} X)$$

where $EG \longrightarrow BG$ is the universal principal G-bundle (Borel construction, Borel cohomology, cf. e.g. [16; III.1]). In this case $H_G^*(\mathrm{pt}) = H^*(BG)$ is the (original non-equivariant) cohomology of the classifying space BG.

From now on in this section and in the next one H^* will always be singular cohomology with coefficients in the prime field $\mathbb{F}_2 = \mathbb{Z}/2$ (which we suppress in the notation), and likewise H_* for homology. The group G will be $O(2)$ or a closed subgroup of it. In particular we have

$$H_{O(2)}^*(\mathrm{pt}) = H^*(BO(2)) = \mathbb{F}_2[w_1, w_2],$$

the graded polynomial algebra over $\mathbb{F}_2$ with the universal Stiefel–Whitney classes w_1, w_2 as generators in degrees (= dimensions) 1 and 2 resp. (cf., e.g., [44; 16.10]).

Assume now the hypotheses of Theorem 1.1. So M is a simply connected closed Riemannian C^∞-manifold such that for some $m \geqslant 2$

$$H_k(M) = H^k(M) = 0 \quad \text{if } 0 < k < m.$$

Furthermore we have a $\mathcal{C}^1$-map $f : S^m \to M$ such that the induced homomorphisms

$$\mathbb{F}_2 \cong H_m(S^m) \xrightarrow{f_*} H_m(M) \quad \text{and} \quad H^m(M) \xrightarrow{f^*} H^m(S^m) \cong \mathbb{F}_2$$

are non-zero.

Consider the (compact) subspace $C \subset \Lambda S^m$ whose elements are the constant (one point) curves and the (ordinary round) circles on S^m parametrized proportionally to arc length. The number $l_{\max}$ in Theorem 1.1 is the maximal value of the length function on the (compact) space $(\Lambda f)(C) \subset \Lambda M$. So in the notation of Sect. 3 we have $(\Lambda f)(C) \subset \Lambda^{l_{\max}} M$.

We are going to use Corollary 3.7 of Theorem 3.5 with this $l_{\max}$ and with $h^* = H_{O(2)}^*$. The task is to find suitable classes $\beta_1, \dots, \beta_r \in H_{O(2)}^*(\Lambda M)$ such that $H_{O(2)}^*(\Lambda M, M) \cdot \beta_1 \cdots \beta_r$ does not restrict to 0 on $(\Lambda^{l_{\max}} M, M)$.

For this let $A \subset C$ be the subspace of all great circles on S^m (parametrized by arc length) which is the same as the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+1}$ and as the space of unit tangent vectors of S^m. We also use the notation $C^l = C \cap \Lambda^l(S^m)$ hence $C^0 = \Lambda^0 S^m = S^m$. The map $p : C - C^0 \longrightarrow A$ which sends each non-constant circle into the great circle parallel to it is a bundle with fibre an open $(m-1)$-disk. It is an $O(2)$-map and $O(2)$ acts freely on A (therefore also on $C - C^0$). Hence, for every closed subgroup G of $O(2)$, passing to orbit spaces gives also an $(m-1)$-disk bundle

$$p/G : (C - C^0)/G \longrightarrow A/G.$$

It has a (unique) Thom class τ_G (remember that we always use mod 2 coefficients) which is an element of $H_G^{m-1}(C - C^0, C^\varepsilon - C^0) = H_G^{m-1}(C, C^\varepsilon) = H_G^{m-1}(C, C^0)$ with $0 < \varepsilon < 2\pi$. In this section we need this only for $G = O(2)$ but in the next one we will use it also for $G = O(1) \cong \mathbb{Z}/2$.

The orbit space $A/O(2)$ is nothing else but the the Grassmannian $G(2, m-1)$ of (non-oriented) planes in $\mathbb{R}^{m+1}$ through the origin. Since $O(2)$ acts freely on A we have

$$H^*_{O(2)}(A) = H^*(A/O(2)) = H^*(G(2, m-1))$$

(cf., e.g., [16; III (1.11)]). This cohomology is known. In terms of the canonical ring homomorphism

$$\mathbb{F}_2[w_1, w_2] = H^*(BO(2)) = H^*_{O(2)}(\mathrm{pt}) \longrightarrow H^*_{O(2)}(A) = H^*(G(2, m-1))$$

and using the letters w_1, w_2 also for the images under this map (cf. beginning of this section) we have

4.1. Proposition. (1) *As a ring with unit, $H^*(G(2, m-1))$ is generated by w_1 and w_2.*

(2) *If q and s are the integers such that $m = 2^q + s$ and $0 \leqslant s < 2^q$ then the product $w_1^{2m-2s-2} w_2^s$ is non-zero in $H^*(G(2, m-1))$ (and it has the maximal number of factors of positive dimension with this property).*

Proof. The first assertion (1) is a result of S.-S. Chern [14; p. 370 Theorem 2] and was also proved (by different methods) by A. Borel [13; Théorème 11.1 and Prop. 11.1]. The second assertion (2), in a somewhat different formulation, goes back to S. I. Al'ber [3; Sect. 13]. Complete descriptions of the relations between w_1 and w_2 are also contained in [14] and [13]. It is not hard to derive (2) from that, and this was done explicitely in [30; 2.3.3]. □

From (2) and the Thom isomorphism for the $(m-1)$-disk bundle

$$p/O(2) : (C - C^0)/O(2) \longrightarrow A/O(2) = G(2, m-1)$$

we obtain

$$\tau_{O(2)} w_1^{2m-2s-2} w_2^s \neq 0 \quad \text{in } H^*_{O(2)}(C, C^0), \tag{4.2}$$

where $\tau_{O(2)} \in H^{m-1}_{O(2)}(C, C^0) \cong H^0(G(2, m-1)) \cong \mathbb{F}_2$ is the Thom class as above.

In Lemma 4.4 we shall prove that $\tau_{O(2)}$ is in the image of $H^{m-1}_{O(2)}(\Lambda M, M)$ under the map induced by

$$(C, C^0) = (C, S^m) \hookrightarrow (\Lambda S^m, S^m) \xrightarrow{\Lambda f} (\Lambda M, M).$$

Therefore (4.2) shows that the restriction of

$$H_{O(2)}(\Lambda M, M) \cdot w_1^{2m-2s-2} w_2^s$$

to $((\Lambda f)(C), (\Lambda f)(C^0))$ and hence to $(\Lambda^{l_{\max}} M, M)$ does not vanish.

The only question left is for which $n = 1, 2 \dots$ are the classes w_1 and w_2 equal to 0 in

$$H^*_{O(2)}(O(2)/R_n) = H^*\big(EO(2) \underset{O(2)}{\times} O(2)/R_n\big) = H^*\big((EO(2))/R_n\big) = H^*(BR_n),$$

where R_n is the rotation group of order n. This is answered by

4.3. Proposition. (1) *For n odd, $H^k(BR_n) = 0$ for all $k > 0$, in particular $w_1 = w_2 = 0$ in $H^*(BR_n)$.*

(2) *For all n one has $w_1 = 0$ in $H^1(BR_n)$.*

Proof. $H^*(BR_n)$ is also the cohomology of the group $R_n \cong \mathbb{Z}/n$ in the algebraic sense. With coefficients mod 2 we obtain (1), e.g., from [33; Chap. IV, Prop. 5.3]. The canonical homomorphism $H^*(BO(2)) \to H^*(BR_n)$ factors through $H^*(BSO(2)) = H^*(\mathbb{C}P^\infty)$ and we get (2) because $H^k(\mathbb{C}P^\infty) = 0$ for all odd k. □

Now we have verified all the hypotheses of 3.6 with

$$r = 2m - s - 2 = m + 2^q - 2 \leqslant 3 \cdot (2^q - 1),$$
$$r_2 = 2(m - s) - 2 = 2 \cdot (2^q - 1).$$

Obviously $r \leqslant 2r_2 + 1$ (since $q \geqslant 1$), so Corollary 3.7 gives the conclusion of Theorem 1.1. □

It remains to prove

4.4. Lemma. *Under the assumptions made on $f : S^m \to M$ the homomorphism*

$$H^{m-1}_G(\Lambda M, M) \longrightarrow H^{m-1}_G(C, S^m) = H^{m-1}_G(C, C^0) \cong H^0(A/G) \cong \mathbb{F}_2$$

induced by Λf is non-zero, and hence surjective, for every closed subgroup G of $O(2)$.

Proof. Let $\tilde{\Lambda} M$ denote the $O(2)$-space of *all continuous* closed curves with the compact-open topology. Since the inclusion of ΛM into $\tilde{\Lambda} M$ is equivariant and continuous [41; Sect. 13], [31; 2.3] (actually the inclusion is even a homotopy equivalence [30; 1.2.10 Theorem] but we do not need that at this point) it suffices to prove that $\tilde{\Lambda} f$ induces a non-zero homomorphism

$$H^{m-1}_G(\tilde{\Lambda} M, M) \longrightarrow H^{m-1}_G(C, S^m).$$

From the hypothesis that M is simply connected and $H^k(M) = 0$ if $0 < k < m$ one obtains canonical isomorphisms

$$H^k_G(\tilde{\Lambda} M, M) \cong H^k(\tilde{\Lambda} M, M) \cong H^k(\Omega M, \mathrm{pt}) \cong H^{k+1}(M) \tag{4.5}$$

for $0 \leqslant k < m$ (implying of course that all of the groups are trivial if $0 \leqslant k < m - 1$) as follows:

The last of these three isomorphisms is a standard application of the Leray–Serre spectral sequence in cohomology mod 2 for the path-loop fibration $\Omega M \hookrightarrow PM \longrightarrow M$ (cf., e.g., [34; Sect. 5.2.2]).

To obtain the other two isomorphisms in (4.5) consider a pair of fibrations

$$\phi : (E, E') \longrightarrow (B, B)$$

with 0-connected base B and fibre pair (F, F'). Then there is a relative version of the Leray–Serre spectral sequence converging to $H^*(E, E')$ such that the E_2-term $E_2^{p,q}$ is p-dimensional (singular) cohomology of B with coefficients in the local system of $\mathbb{F}_2$-vectorspaces formed by $H^q(F, F')$ with the usual operation of $\pi_1(B)$, cf., e.g., [34; Chap. 5] together with [44; Remark 2 on p. 351]. In particular, $E_2^{0,q}$ is the fixed point subvectorspace $H^q(F, F')^{\pi_1(B)}$ of $H^q(F, F')$ under the action of $\pi_1(B)$. If, in addition, $H^q(F, F') = 0$ for all $q < k$, then the inclusion $(F, F') \hookrightarrow (E, E')$ induces the isomorphism $H^k(E, E') \cong E_\infty^{0,k} = E_2^{0,k} = H^k(F, F')^{\pi_1(B)}$. Now we obtain the second isomorphism in (4.5) by taking

$$B = M, \quad (E, E') = (\widetilde{\Lambda} M, M), \quad \phi(\gamma) = \gamma(1) \text{ for all } \gamma \in \widetilde{\Lambda} M.$$

Here $\pi_1(B) = \pi_1(M) = 0$, hence the local coefficient system is simple.

For the first isomorphism in (4.5) we take

$$\begin{aligned} &B = BG, \qquad (E, E') = (EG \times_G \widetilde{\Lambda} M, BG \times M), \\ &\phi : EG \times_G \widetilde{\Lambda} M \longrightarrow BG \text{ the canonical projection.} \end{aligned}$$

In this case the local coefficient system might not be simple. The action of $\pi_1(B) = \pi_1(BG) = \pi_0(G)$ on the cohomology of the fibre pair $(\widetilde{\Lambda} M, M)$ is induced by the standard $O(2)$-action on $\widetilde{\Lambda} M$. Since $\pi_0(O(1)) \cong \pi_0(O(2))$ we need only look at the $O(1)$-action. But the fibrations giving the second and third isomorphisms in (4.5) are $O(1)$-equivariant and $O(1)$ acts trivially on $M = \Lambda^0 M$. Hence $\pi_1(B) = \pi_0(G)$ acts trivially on $H^k(\widetilde{\Lambda} M, M)$ for $0 \leqslant k < m$.

The isomorphisms (4.5) are natural with respect to $f : S^m \to M$. Hence, in order to prove Lemma 4.4, it is enough to show that the inclusion $j : C \hookrightarrow \widetilde{\Lambda} S^m$ induces a non-zero homomorphism $j^* : H^{m-1}(\widetilde{\Lambda} S^m, S^m) \longrightarrow H^{m-1}(C, S^m)$. Define $g : (D^{m-1}, S^{m-2}) \longrightarrow (C, S^m)$ (where D^{m-1} is the closed unit disk in $\mathbb{R}^{m-1}$) by

$$g(x)(e^{it}) = (x, \sqrt{1 - \|x\|^2} \cdot \cos t, \sqrt{1 - \|x\|^2} \cdot \sin t).$$

Denoting by D_+^m the upper hemisphere of S^m, there is a unique map h making the following diagram commutative

$$\begin{array}{ccccc} (D^{m-1}, S^{m-2}) & \xrightarrow{g} & (C, S^m) & \xhookrightarrow{j} & (\widetilde{\Lambda} S^m, S^m) \\ \downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \widetilde{\Lambda}\psi} \\ (S^{m-1}, \mathrm{pt}) & \xrightarrow{h} & \Omega(S^m/D_+^m, \mathrm{pt}) & \xhookrightarrow{i} & (\widetilde{\Lambda}(S^m/D_+^m), S^m/D_+^m), \end{array} \tag{4.6}$$

where φ and ψ are the canonical projections onto quotient spaces. Obviously φ induces an isomorphism in (co)homology. The adjoint of h,

$$S^m \cong \Sigma S^{m-1} \longrightarrow S^m/D_+^m \cong S^m$$

(Σ denoting reduced suspension) is a map of degree ± 1, hence h induces an isomorphism in H^{m-1} by the (co)homology suspension theorem (cf., e.g., [44; 15.45]). By (4.5) the inclusion i also induces an isomorphism in H^{m-1}. Now the commutativity of (4.6) tells us that j^* is indeed non-zero in H^{m-1}. $\square$

5. Proof of Theorem 1.3 (Projective Spaces) including Supplement 1.7

As in the proof of Theorem 1.1 in the previous section we will apply Corollary 3.7, this time however not only with Borel cohomology for the group $O(2)$ but (following Rademacher [42]) also for $O(1) \cong \mathbb{Z}/2$ which we embed into $O(2)$ by

$$\varepsilon \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

Assume we have a simply connected closed Riemannian $\mathcal{C}^\infty$-manifold M and a $\mathcal{C}^1$-map $f : \mathbb{F}P^m \longrightarrow M$, where $\mathbb{F}P^m$ is the projective space over $\mathbb{F}$, which stands for the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers Ca ($m \leqslant 2$ in this last case). For $d = \dim_{\mathbb{R}} \mathbb{F}$ ($= 2, 4$ or 8) assume that

$$\begin{aligned} &H^k(M) = 0 \quad \text{for } 0 < k < d, \quad \text{and} \\ &f^* : H^d(M) \longrightarrow H^d(\mathbb{F}P^m) \cong \mathbb{F}_2 \quad \text{is non-zero, hence surjective,} \end{aligned} \tag{5.1}$$

where like in the previous section H^* is singular cohomology with mod 2 coefficients. This is the same hypothesis as (1.8) in Supplement 1.7. In Theorem 1.3 itself we made the stronger hypothesis that f induces an isomorphism in all of H^*.

We proceed similarly to Sect. 4. This time $C \subset \Lambda(\mathbb{F}P^m)$ is the compact space of those closed curves on $\mathbb{F}P^m$ which are one point curves or circles in the sense of Sect. 4 on one of the projective lines in $\mathbb{F}P^m$. Remember that $\mathbb{F}P^m$ has a canonical metric such that each of these lines is isometric to the standard d-sphere (Sect. 1 and [11; Chap. 3]). Again $l_{\max}$ is the maximal value of the length function on $(\Lambda f)(C)$.

By A we denote the subspace of C consisting of all great circles on one of the projective lines and with $C^l := C \cap \Lambda^l(\mathbb{F}P^m)$ we define $p : C - C^0 \longrightarrow A$ by sending every non-constant circle to the great circle which lies on the same projective line and is parallel to it. The map p is an $O(2)$-equivariant bundle with fibre an open $(d-1)$-disk and for every closed subgroup G of $O(2)$ it induces

$$p/G : (C - C^0)/G \longrightarrow A/G,$$

which is again a bundle with fibre an open $(d-1)$-disk (because $O(2)$ acts freely on A). Let $\tau_G \in H_G^{d-1}(C, C^0)$ be its Thom class (details as in Sect. 4).

5.2. Proposition. *Under the assumptions made on $f : \mathbb{F}P^m \to M$, the homomorphism $H_G^{d-1}(\Lambda M, M) \longrightarrow H_G^{d-1}(C, C^0) \cong H^0(A/G) \cong \mathbb{F}_2$ induced by*

$$(C, C^0) = (C, \mathbb{F}P^m) \hookrightarrow (\Lambda(\mathbb{F}P^m), \mathbb{F}P^m) \xrightarrow{\Lambda f} (\Lambda M, M)$$

is non-zero, hence surjective, for every closed subgroup G of $O(2)$.

Proof. If L is a projective line in FP^m then the composition

$$S^d \cong L \hookrightarrow FP^m \xrightarrow{f} M$$

satisfies the hypotheses of Lemma 4.4 with d instead of m. Hence even if we restrict Λf to $(C \cap \Lambda L, L)$ we get a non zero homomorphism in H_G^{d-1}, the more so if we restrict only to $(C, C^0) = (C, \mathbb{F}P^m)$. $\square$

Now we consider the case $G = O(1)$ and the corresponding Borel cohomology theory $H^*_{O(1)}$ with mod 2 coefficients. Its coefficient algebra $H^*_{O(1)}(\mathrm{pt}) = H^*(BO(1)) = H^*(B(\mathbb{Z}/2)) = H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[w_1]$ is the polynominal algebra generated by the first Stiefel–Whitney class $w_1 \in H^1(BO(1))$.

The base-point map $\varphi_M : \Lambda M \to M$ which associates to each closed curve $\gamma \in \Lambda M$ its base-point $\gamma(1)$ is $O(1)$-invariant and gives rise to the following commutative diagram of $O(1)$-equivariant maps

$$\begin{tikzcd} A \arrow[r, hook] & C \arrow[r, hook] & \Lambda\mathbb{F}P^m \arrow[r, "\Lambda f"] \arrow[d, "\phi_{\mathbb{F}P^m}"'] & \Lambda M \arrow[d, "\phi_M"] \\ & & \mathbb{F}P^m \arrow[r, "f"] & M. \end{tikzcd}$$

Let $c \in H^d(\mathbb{F}P^m)$ be the generator. By assumption there is an element $z \in H^d(M)$ with $f^*z = c$. Let $\tilde{z}$ be the image of $1 \otimes z$ under the map

$$H^*(BO(1)) \otimes H^*(M) = H^*(BO(1) \times M) = H^*_{O(1)}(M) \xrightarrow{\varphi^*_M} H^*_{O(1)}(\Lambda M).$$

Similarly we construct $\tilde{c}$ from c. Then $(\Lambda f)^*\tilde{z} = \tilde{c}$ by the diagram above. Rademacher has shown

5.3. Proposition [42; 4.1]. *For the integers q and s as in Proposition* 4.1, *i.e., characterized by $m = 2^q + s$ and $0 \leqslant s < 2^q$, the product*

$$w_1^{2dm-2ds-d-1}(\tilde{c}|A)^{2s+1}$$

is non-zero in $H^*_{O(1)}(A)$ (and it has the maximal number of factors of positive dimension with this property). □

By the Thom isomorphism we get that

$$\tau_{O(1)} w_1^{2dm-2ds-d-1}(\tilde{c}|C)^{2s+1} \neq 0 \quad \text{in } H^*_{O(1)}(C, C^0), \tag{5.4}$$

where $\tau_{O(1)}$ is the Thom class of the bundle $p/O(1) : (C - C^0)/O(1) \longrightarrow A/O(1)$. By Proposition 5.2 this Thom class $\tau_{O(1)}$ is in the image of $H^{d-1}_{O(1)}(\Lambda M, M)$. Therefore (5.4) shows that the restriction of $H^*_{O(1)}(\Lambda M, M) \cdot w_1^{2dm-2ds-d-1}\tilde{z}^{2s+1}$ to $((\Lambda f)(C), (\Lambda f)(C) \cap M)$ and hence to $(\Lambda^{l_{\max}} M, M)$ does not vanish. Both factors w_1 and $\tilde{z}$ restrict to 0 on

$$H^*_{O(1)}(O(2)/R_n) = H^*\big(O(2)/(O(1)\cdot R_n)\big) = H^*(SO(2)/R_n) = H^*(S^1)$$

for all n and under any $O(2)$-equivariant map $O(2)/R_n \longrightarrow \Lambda M$. This is clear for $\tilde{z}$ because its dimension is $d > 1$, and it follows for w_1 from the factorization of $O(2)$-maps

$$O(2)/R_n \longrightarrow O(2)/SO(2) = O(1) \longrightarrow \mathrm{pt}$$

and the fact that $H^*_{O(1)}(O(1)) = H^*(\mathrm{pt})$.

Thus we have verified all the hypotheses of 3.6 with

$$r = r_2 = 2dm - 2(d-1)s - d.$$

The number $r+1$ is nothing else but $g_d^R(m)$ (1.6) and it equals the number $g_d(m)$ used in Theorem 1.3 if

$$d=2 \qquad \text{or} \qquad d=4 \text{ and } s>1.$$

So Corollary 3.7 (even a particularly simple version of it in which the distinction between r and r_2 plays no rôle) gives the conclusion of Theorem 1.3 under the weakened hypothesis (1.8) = (5.1) if *either* $g_d(m)$ is replaced by $g_d^R(m)$ *or* one excludes the cases

$$\begin{array}{lll} \mathbb{F}=\mathbb{H}, & d=4 \text{ and } s=0,1, & \\ \mathbb{F}=\mathrm{Ca}, & d=8 \text{ and } m=1,2 & (\text{in particular } s=0). \end{array}$$

In order to obtain the full result also in these cases (where $g_d^R(m)$ is smaller than $g_d(m)$) we return to $G=O(2)$ and we use $O(2)$-Borel cohomology $H^*_{O(2)}$ (again with $\mathbb{F}_2$-coefficients).

First we settle the case $\mathbb{F}=\mathrm{Ca}$, $d=8$, $m=1$ or 2. (Actually one could always restrict to $m \geqslant 2$ because for $m=1$ Theorem 1.3 together with Supplement 1.7 is the same as Theorem 1.1 with d instead of m.) Klingenberg has shown in [29; 4.2] that $w_1^{16m-2} \neq 0$ in $H^*_{O(2)}(A)$. The Thom isomorphism for the bundle

$$p/O(2):(C-C^0)/O(2) \longrightarrow A/O(2)$$

gives $\tau_{O(2)} w_1^{16m-2} \neq 0$ in $H^*_{O(2)}(C, C^0)$ like in (4.2) and (5.4). Using Proposition 5.2 we conclude that the restriction of

$$H^*_{O(2)}(\Lambda M, M)\cdot w_1^{16m-2}$$

to $((\Lambda f)(C), (\Lambda f)(C)\cap M)$ and hence to $(\Lambda^{l_{\max}} M, M)$ does not vanish. We know already from 4.3(2) that $w_1=0$ in $H^1(BR_n)$ for every n. Thus we have verified all the hypotheses of 3.6 with

$$r=r_2=16m-2=g_8^K(m)-1=g_8(m)-1$$

and 3.7 gives Theorem 1.3 including Supplement 1.7 in the case we are considering ($\mathbb{F}=\mathrm{Ca}$). Note that $g_8^R(m)$ is only $16m-7$, so that the first method based on $O(1)$-cohomology gives a much weaker result.

Now only the case $d=4$, $s=0$ or 1 is left. But for the time being we allow $d=2$ or 4 and s arbitrary. The map $q_d: A \longrightarrow G_d(2, m-1)$ into the Grassmannian $G_d(2, m-1)$ of 2-dimensional $\mathbb{F}$-subvectorspaces of $\mathbb{F}^{m+1}$ which sends each parametrized great circle into the projective line it lies on is $O(2)$-invariant, hence it induces a map

$$q_d/O(2): A/O(2) \longrightarrow G_d(2, m-1)$$

(which is actually a fibre bundle with fibre the real Grassmannian $G(2, d-1)$).

The $\mathbb{F}_2$-algebra $H^*(G_d(2, m-1))$ is generated by two classes c_1 and c_2 in dimensions d and $2d$ resp. For $\mathbb{F}=\mathbb{C}$ these are the Chern classes mod 2 of the canonical $U(2)$-bundle over $G_d(2, m-1)$ and for $\mathbb{F}=\mathbb{H}$ the Pontryagin classes mod 2 of the canonical $Sp(2)$-bundle. This is analogous to 4.1(1) and with the method of A. Borel it can be proved in precisely the same way as 4.1(1) [13; Remark after Th. 12.1].

Let $\hat{c}_i \in H^{id}_{O(2)}(A) = H^{id}(A/O(2))$ be the image of c_i under $(q_d/O(2))^*$, $i=1,2$. Klingenberg has shown

5.5. Proposition [29; 4.3]. (1) *As a ring with unit, $H^*_{O(2)}(A) = H^*(A/O(2))$ is generated by the classes w_1, w_2, $\hat{c}_1$ and $\hat{c}_2$.*

(2) *For the integers q and s as above, i.e., characterized by $m = 2^q + s$ and $0 \leqslant s < 2^q$, the product $w_1^{2dm-2ds-2}\hat{c}_2^s$ is non-zero in $H^*_{O(2)}(A) = H^*(A/O(2))$ and it has the maximal number of factors of positive dimension with this property.* □

We are going to show below

5.6. Lemma. *Let $f : \mathbb{F}P^m \to M$ be a map which induces isomorphisms in H^k for all $k \leqslant 2d$. Then there is an element $x \in H^{2d}_{O(2)}(\Lambda M)$ such that $\hat{c}_2 - ((\Lambda f)|A)^*x$ is contained in the ideal of $H^*_{O(2)}(A) = H^*(A/O(2))$ generated by the classes w_1, w_2 and $\hat{c}_1$.*

Looking into the details of [29] one can show

5.7. Lemma. *The product of $w_1^{2dm-2ds-2}$ with any of the classes w_1, w_2 and $\hat{c}_1$ vanishes in $H^*_{O(2)}(A) = H^*(A/O(2))$.*

In the cases which are relevant for us we shall give the proof below. It follows that

$$w_1^{2dm-2ds-2}\big(((\Lambda f)|A)^*x\big)^s = w_1^{2dm-2ds-2}\hat{c}_2^s \neq 0 \quad \text{in } H^*_{O(2)}(A) = H^*(A/O(2)),$$

and by the Thom isomorphism that $\tau_{O(2)}w_1^{2dm-2ds-2}\big(((\Lambda f)|A)^*x\big)^s \neq 0$ in $H^*_{O(2)}(C, C^0)$.

Using Proposition 5.2 we conclude that the restriction of $H^*_{O(2)}(\Lambda M, M) \cdot w_1^{2dm-2ds-2}x^s$ to $((\Lambda f)(C), (\Lambda f)(C) \cap M)$ and hence to $(\Lambda^{l_{\max}}M, M)$ does not vanish.

From Proposition 4.3 we know that w_1 restricts to 0 on $O(2)/R_n$ for any n, whereas x does so for odd n. Thus we have verified all the hypotheses of 3.6 with

$$r = 2dm - (2d-1)s - 2, \quad r_2 = 2dm - 2ds - 2.$$

One easily checks that the additional condition $r \leqslant 2r_2 + 1$ of 3.7 is always satisfied. Hence we get the conclusions of Theorem 1.3 for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$ with

$$r + 1 = 2dm - (2d-1)s - 1 = m + (2d-1)2^q - 1 = g_d^K(m)$$

(compare (1.5)) instead of $g_d(m)$. If $s = 0$, Lemma 5.6 is not needed, so the hypothesis (5.1) on the $\mathcal{C}^1$-map $f : \mathbb{F}P^m \longrightarrow M$ suffices. Otherwise, as in Lemma 5.6, we have to assume that f induces isomorphisms in H^k for all $k \leqslant 2d$.

This completes the proof of Theorem 1.3 including Supplement 1.7 because in the only case which was left ($d = 4$, $s = 0$ or 1) the numbers $g_4^K(m)$ and $g_4(m)$ coincide. □

We still have to prove Lemmas 5.6 and 5.7 (at least in the case $d = 4$, $s = 1$). Before we do so let us compare the strength of the two methods we used: $O(1)$-cohomology following Rademacher giving $g_d^R(m)$ and $O(2)$-cohomology following Klingenberg/Hingston giving $g_d^K(m)$. Our number $g_d(m)$ is the maximum of the two. Rademacher's number is larger than Klingenberg's if $d = 2, s > 0$ or $d = 4, s > 2$, the opposite is true if $d = 4, s < 2$ or $d = 8$, and the two numbers are equal if $d = 2, s = 0$ or $d = 4, s = 2$. The hypothesis (1.8)=(5.1) on f is sufficient for the $O(1)$-method in all cases, and for the $O(2)$-method in the case $s = 0$; otherwise one needs the stronger hypothesis that f induces isomorphisms in H^k for all $k \leqslant 2d$.

Proof of 5.7 for $s \leqslant 1$. The result is trivial (and uninteresting) for $s = 0$ because $2dm - 2$ is the dimension of $A/O(2)$. Let $s = 1$ and let x be one of the classes w_1, w_2 and $\hat{c}_1$. Then x has dimension 1, 2 or d resp., in any case $< 2d$. If $w_1^{2dm-2d-2}x \neq 0$ then by Poincaré duality for the manifold $A/O(2)$ there would exist some $y \in H^*(A/O(2))$ of *positive dimension* such that $w_1^{2dm-2d-2}xy$ is the generator of the top dimensional cohomology group $H^{2dm-2}(A/O(2))$. This contradicts the maximality assertion in Proposition 5.5. □

Proof of 5.6. If $m \leqslant 2$ we have $c_2 = 0$ in $H^{2d}(G_d(2, m-1))$, because then $G_d(2, m-1)$ is a point ($m = 1$) or the projective plane over $\mathbb{F}$ ($m = 2$). Hence we assume $m \geqslant 3$. As in the proof of Lemma 4.4 we may replace the functor Λ by its "topological" counterpart $\widetilde{\Lambda}$ (the space of all continuous closed curves with the compact-open topology). Let us look at the commutative diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[drr, "\varphi"'] & C \arrow[r, hook] & \widetilde{\Lambda}\mathbb{F}P^m \arrow[r, "\widetilde{\Lambda}f"] \arrow[d, "\widetilde{\varphi}_{\mathbb{F}P^m}"] & \widetilde{\Lambda}M \arrow[d, "\widetilde{\varphi}_M"] \\ & & \mathbb{F}P^m \arrow[r, "f"] & M, \end{tikzcd}$$

where $\widetilde{\varphi}_M$ is the base point map sending each $\gamma \in \widetilde{\Lambda}M$ into its starting point $\gamma(1)$. The restriction φ of $\widetilde{\varphi}_{\mathbb{F}P^m}$ to the space A of great circles on $\mathbb{F}P^m$ is a bundle with fibre S^{dm-1}. Since $m \geqslant 3$ and $d \geqslant 2$, this implies that φ induces an isomorphism in H^k for all $k \leqslant 2d$. Consider the Borel fibration

$$A \overset{i}{\hookrightarrow} EO(2) \underset{O(2)}{\times} A \xrightarrow{\pi} BO(2).$$

The map i induces the canonical homomorphism $i^* : H^*_{O(2)}(A) \longrightarrow H^*(A)$. We claim that it sends w_1, w_2 and $\hat{c}_1$ to zero, and $\hat{c}_2$ to $(\varphi^*c)^2 \neq 0$ where $c \in H^d(\mathbb{F}P^m)$ is the generator. This is obvious for the classes w_1, w_2 since they are in the image of π^*. To prove it for $\hat{c}_1$ and $\hat{c}_2$, observe firs .iat we may replace m by ∞. Then, up to homotopy, q_d factors as

$$q_d : A \xrightarrow{\varphi} \mathbb{F}P^\infty \xrightarrow{\text{diag}} \mathbb{F}P^\infty \times \mathbb{F}P^\infty \xrightarrow{g} G_d(2, \infty),$$

where g is the classifying map of $\eta \times \eta$, η being the canonical line bundle. It is well known that $g^*(c_1) = c \otimes 1 + 1 \otimes c$, $g^*(c_2) = c \otimes c$ [44; 16.10–16.13], hence

$$\pi^*(\hat{c}_1) = q_d^*(c_1) = 2 \cdot \varphi^*c, \quad \pi^*(\hat{c}_2) = q_d^*(c_2) = (\varphi^*c)^2$$

(compare [42; Sect. 3]). It follows that the kernel of $i^* : H^{2d}_{O(2)}(A) \longrightarrow H^{2d}(A)$ is contained in the ideal generated by w_1, w_2 and $\hat{c}_1$. By hypothesis there is an element $z \in H^d(M)$ with $f^*(z) = c$. Let $y = \widetilde{\varphi}_M^*(z) \in H^d(\widetilde{\Lambda}M)$. Then all we need to show is that y^2 is in the image of $H^{2d}_{O(2)}(\widetilde{\Lambda}M) \longrightarrow H^{2d}(\widetilde{\Lambda}M)$. We do this explicitely for $d = 4$ which, as we mentioned above, is the only relevant case. The proof for $d = 2$ is completely analogous.

We will have to do calculations in the Leray–Serre spectral sequences of various fibrations. A general reference for this is the book by J. McCleary [34]. First we need information on the cohomology of the space ΩM of (pointed) loops on M. We obtain it from the path-loop fibration

$$\Omega M \hookrightarrow PM \longrightarrow M$$

[34; Sect. 5.2.2], whose E_2-term is $E_2^{p,q} = H^p(M) \otimes H^q(\Omega M)$. Since PM is acyclic, the only part of this which survives to E_∞ is $E_2^{0,0}$. Using the known structure of $H^p(M)$ for $p \leqslant 8$ one sees immediately that $H^q(\Omega M) = 0$ for $q = 1, 2, 4, 5$, whereas $H^3(\Omega M) = E_2^{0,3} = E_4^{0,3}$ contains precisely one non-zero element v and $d_4(v) = z$ in $H^4(M) = E_2^{4,0} = E_4^{4,0}$. This implies $d_4(z \otimes v) = z^2 \neq 0$ in $H^8(M) = E_2^{8,0} = E_4^{8,0}$ and hence $H^q(\Omega M) = 0$ for $q = 6, 7$.

Next we look at the base-point fibration

$$\Omega M \hookrightarrow \widetilde{\Lambda} M \longrightarrow M,$$

which has the same E_2-term as the path-loop fibration above, but quite different differentials. Using our knowledge of $H^k(M)$ and $H^k(\Omega M)$ for all $k \leqslant 7$ and the fact that this fibration has a section one easily sees that

$$H^q(\widetilde{\Lambda} M) = \begin{cases} 0 & \text{if } q = 1, 2, 5, 6, \\ \mathbb{F}_2 & \text{if } q = 3, 4, 7 \end{cases}$$

and that the only non-zero elements in these groups are y in dimension 4, an element $\tilde{v}$ in dimension 3 which restricts to $v \in H^3(\Omega M)$, and $y\tilde{v}$ in dimension 7.

Now we turn to the Borel fibration

$$\widetilde{\Lambda} M \overset{i}{\hookrightarrow} EO(2) \underset{O(2)}{\times} \widetilde{\Lambda} M \longrightarrow BO(2), \tag{$\mathcal{F}$}$$

for which $E_2^{p,q}$ is p-dimensional cohomology of $BO(2)$ with coefficients in a local system formed by $H^q(\widetilde{\Lambda} M)$ with an action of $\mathbb{Z}/2$ (the fundamental group of the base $BO(2)$). This action is trivial for $q \leqslant 7$ since in this range each of the groups $H^q(\widetilde{\Lambda} M)$ contains at most one non-zero element. We do not know this for $q = 8$, but what we do know is that $y^2 \in H^8(\widetilde{\Lambda} M)$ is fixed under the action (because y is) and hence lies in $E_2^{0,8}$. We are going to show that it is in the image of i^* by checking that it is a permanent cycle in the spectral sequence belonging to ($\mathcal{F}$), i.e., $d_r(y^2) = 0$ for every differential d_r, $r \geqslant 2$. Since we are in characteristic 2 and d_r is a derivation, we immediately have $d_2(y^2) = 0$. The rest follows by induction from

5.8. Assertion. *For all $r \geqslant 3$, if y^2 survives to E_r (i.e., $d_s(y^2) = 0$ for $2 \leqslant s < r$) then $d_r(y^2) = 0$ in $E_r^{r,9-r}$.*

Proof. For $q \leqslant 7$ we explicitly know the structure of $E_2^{*,q} = H^*(BO(2)) \otimes H^q(\widetilde{\Lambda} M) = \mathbb{F}[w_1, w_2] \otimes H^q(\widetilde{\Lambda} M)$, in particular, $E_2^{*,q} = 0$ if $q = 1, 2, 5, 6$, hence 5.8 is obvious except if $r = 5, 6$ or 9. For $r = 9$ it follows from the fact that the fibration ($\mathcal{F}$) has a section ($\widetilde{\Lambda} M$ has fixed points), so that all of $E_2^{*,0} = H^*(BO(2))$ survives to E_∞ and cannot contain any non-zero d_r-boundaries. What remains are the cases $r = 5, 6$:

Suppose for a moment that $y \in H^4(\widetilde{\Lambda} M) = E_2^{0,4}$ is a permanent cycle (actually this is not the case but we need not decide it here). Then $y^2 \in H^8(\widetilde{\Lambda} M)$ would also be a permanent cycle (because each d_r is a derivation). So we may assume that y is *not* a permanent cycle. The only possibility for this to happen is that $d_2(y) = u_2 \otimes \tilde{v}$ for some non-vanishing $u_2 \in H^2(BO(2))$ (it will turn out below that in fact $u_2 = w_2$, but we don't care at the moment). Every element of $E_2^{*,4}$ has the form $u \otimes y$ with $u \in H^*(BO(2)) = \mathbb{F}[w_1, w_2]$. Since d_2 is a derivation we get

$$d_2(u \otimes y) = uu_2 \otimes \tilde{v} \quad \text{in } E_2^{*,3} = \mathbb{F}[w_1, w_2] \otimes H^3(\widetilde{\Lambda} M), \tag{5.9}$$

which is non-zero because $\mathbb{F}[w_1, w_2]$ has no zero divisors. Hence $E_r^{p,4} = 0$ for all $p \geqslant 0, r \geqslant 3$ and therefore $d_5(y^2) \in E_5^{5,4}$ vanishes.

To settle the last case $r = 6$ we compare the fibration $(\mathcal{F})$ to the corresponding one for the group $O(1)$ instead of $O(2)$

$$\widetilde{\Lambda} M \hookrightarrow EO(1) \underset{O(1)}{\times} \widetilde{\Lambda} M \longrightarrow BO(1), \qquad (\bar{\mathcal{F}})$$

for which $\bar{E}_2^{p,q}$ is p-dimensional cohomology of $BO(1)$ with coefficients in a local system formed by $H^q(\widetilde{\Lambda} M)$ which as in the case of $(\mathcal{F})$ is trivial for $q \leqslant 7$. The inclusion $j : O(1) \hookrightarrow O(2)$ induces a map of fibrations $(\bar{\mathcal{F}}) \to (\mathcal{F})$ and hence a map of the spectral sequences in the opposite direction $j^* : E_r \to \bar{E}_r$. For the E_2-terms, as long as $q \leqslant 7$, we get the canonical epimorphism

$$\mathbb{F}[w_1, w_2] \otimes H^q(\widetilde{\Lambda} M) = E_2^{*,q} \longrightarrow \bar{E}_2^{*,q} = \mathbb{F}[w_1] \otimes H^q(\widetilde{\Lambda} M)$$

whose kernel is the ideal generated by w_2. On the other hand we compare $(\bar{\mathcal{F}})$ to the trivial fibration

$$M \hookrightarrow EO(1) \underset{O(1)}{\times} M = BO(1) \times M \longrightarrow BO(1). \qquad (\mathcal{F}_0)$$

Since the base-point map $\widetilde{\varphi}_M : \widetilde{\Lambda} M \to M$ is $O(1)$-invariant, it induces a map of fibrations $\bar{\mathcal{F}} \to \mathcal{F}_0$ and hence a map of the spectral sequences in the opposite direction (which actually is an embedding because $\widetilde{\varphi}_M$ has a section).

Remember that $y = \widetilde{\varphi}_M^*(z)$ where $z \in H^4(M)$. In the spectral sequence associated to $(\mathcal{F}_0)$ all differentials vanish since the fibration is trivial. Hence in the spectral sequence $\bar{E}_r$ associated to $(\bar{\mathcal{F}})$ all elements coming from $(\mathcal{F}_0)$ are permanent cycles, in particular $y \in \bar{E}_r^{0,*}$ and all its powers. For the spectral sequence E_r associated to $(\mathcal{F})$ this implies $d_2 y = w_2 \otimes \tilde{v}$ since the latter is the only non-zero element in the kernel of $j^* : E_2^{2,3} \to \bar{E}_2^{2,3}$. So we know now that $u_2 = w_2$ in (5.9), and then (5.9) implies that j induces an isomorphism $E_3^{*,3} \cong \bar{E}_3^{*,3}$.

The sum of $\bar{E}_2^{*,q}$ for $q \leqslant 7$ is multiplicatively generated by $w_1 \in \bar{E}_2^{1,0}$, $\tilde{v} \in H^3(\widetilde{\Lambda} M) = \bar{E}_2^{0,3}$, and $y \in H^4(\widetilde{\Lambda} M) = \bar{E}_2^{0,4}$. All of these are permanent cycles. We just saw this for y, and using the fact that $(\bar{\mathcal{F}})$ has a section it is obvious for the other two. It follows that for $q \leqslant 7$ all of $\bar{E}_2^{*,q}$ consists of permanent cycles. This implies the injectivity of

$$j^* : E_6^{*,3} \longrightarrow \bar{E}_6^{*,3} = \bar{E}_5^{*,3} = \bar{E}_4^{*,3} = \bar{E}_3^{*,3}$$

and from $d_6(y^2) = 0$ in $\bar{E}_3^{6,3}$ we obtain $d_6(y^2) = 0$ in $E_3^{6,3}$. $\square$

5.10. Remark. The proof of Lemma 5.6 fills in a gap in N. Hingston's paper [25; Lemma in 4.3]. There it is argued by saying that "the transgression of a square is zero" which, in general, is not true.

6. Concluding remarks

No example of a closed Riemannian manifold M of dimension $\geqslant 2$ is known to have only finitely many prime closed geodesics. So one could conjecture that every such M has infinitely many of them. We are going to give a survey on the cases for which this is known to be true and then we will discuss the question to what extent our results contribute to the remaining ones. To keep the discussion within reasonable bounds we will only look at conditions which are invariant under homotopy equivalences and we restrict ourselves to simply connected closed manifolds M. A comprehensive survey on closed geodesics is [9]. For the contents of this section we are indebted to communications from Hans-Werner Henn, Matthias Kreck and Thomas Morgenstern.

In order to avoid repetitions let us agree that in this section the term *sufficient condition* will always mean a condition on the homotopy type of a simply connected closed Riemannian manifold M which ensures the existence of infinitely many prime closed geodesics. The most recent result of this kind (mentioned already in the introduction) is

6.1. Sufficient Condition. *M has dimension 2, hence, being closed and simply connected, it is diffeomorphic to the 2-sphere S^2.*

This was proved in [21] combined with [10]. It seems that all other known sufficient conditions in the sense just defined are consequences of

6.2. Sufficient Condition. *The Betti numbers of the free loop space of M with respect to some field F of coefficients form an unbounded sequence.*

This is the celebrated theorem of D. Gromoll and W. Meyer. Strictly speaking they proved it only for $F = \mathbb{Q}$ [22; Theorem 4], but almost the same proof gives the general result which can be found, e.g., in [30; 4.2.9]. In the formulation of 6.2 it is irrelevant whether "free loop space" means the Hilbert manifold ΛM or the space $\tilde{\Lambda} M$ of all continuous maps $S^1 \to M$ with the compact-open topology since these two spaces have the same homotopy type [30; Theorem 1.2.10].

Condition 6.2 becomes particularly interesting in connection with

6.3. Theorem [45]. *For any simply connected space X, the Betti numbers of the free loop space of X with respect to the field $\mathbb{Q}$ of rational numbers form an unbounded sequence if and only if the cohomology algebra $H^*(X; \mathbb{Q})$ is not monogenic, i.e., it cannot be generated (as a $\mathbb{Q}$-algebra with unit) by one single element.*

An immediate corollary is

6.4. Sufficient Condition. *The cohomology algebra $H^*(M; \mathbb{Q})$ of M with rational coefficients is not monogenic.*

This includes a lot of interesting manifolds, e.g., all non-trivial cartesian products. There are, however, also many manifolds whose rational cohomology algebra *is* monogenic.

Unfortunately, it is not known whether the analogue of 6.3 for fields other than $\mathbb{Q}$ holds. For a discussion of this and partial results cf. [35, in particular Corollary 3.4].

The only further sufficient condition (in our sense) we know of is

6.5. Sufficient Condition. *The manifold M has the 2-local homotopy type of a compact, simply connected, homogeneous space (quotient of two Lie groups) which is not diffeomorphic to one of the manifolds listed in* (0.1).

This follows from [36], [37] (except for $M = S^2$, where we have 6.1). The main theorem in [36] says that the mod 2 Betti numbers of the free loop space of M form an unbounded sequence (so that 6.2 is applicable) if M is a homogeneous space which is not diffeomorphic to a symmetric space of rank 1. In dimensions $\geqslant 3$, the manifolds listed in (0.1) are exactly those which can carry the structure of a symmetric space of rank 1. By definition two simply connected CW-complexes X and Y have the same *2-local homotopy type* iff their localizations at the prime 2 [24; Chap. II, Sect. 1] have the same (ordinary) homotopy type. The localization map $X \to X_{(2)}$ at the prime 2 induces an isomorphism in mod 2 homology [24; Chap. II, Proof of Theorem 1.14]. Hence it also induces an isomorphism in mod 2 homology for the free loop spaces as one can easily see by comparing the spectral sequences of the path-loop fibration $\Omega M \hookrightarrow PM \longrightarrow M$ and the base-point fibration $\Omega M \hookrightarrow \widetilde{\Lambda M} \longrightarrow M$ with the corresponding ones for $X_{(2)}$ (cf. also the proofs of 4.4 and 5.6). It follows that the mod 2 Betti numbers of the free loop space are invariants of the 2-local homotopy type. □

How much do our results 1.1–1.4 say about the class of (simply connected closed) manifolds not covered by the preceding "sufficient conditions"?

Observe first that 1.1 (and hence 1.2) are always applicable: If X is any simply connected space with non-trivial $H_*(X; \mathbb{F}_2)$, let $m > 0$ be minimal with $H_m(X; \mathbb{F}_2) \neq 0$. By the Hurewicz-Serre Theorem modulo the class of abelian groups consisting only of torsion elements of odd order (cf., e.g., [43; Chap. 9 Sect. 6 Theorem 15]) there exists a map $f : S^m \to X$ which is non-trivial in $H_m(\,\cdot\,; \mathbb{F}_2)$. If, in addition, $X = M$ is a closed differentiable manifold we can choose f to be a $\mathcal{C}^1$-map. Then all the hypotheses of Theorem 1.1 are satisfied and we get some information on the number of closed geodesics on M. In any case we recover the Lusternik–Fet Theorem giving at least one closed geodesic. How many more we get will depend on the particular numbers m, $l_{\max}$ and $l_{\min}$.

Let us assume now that $H^*(M; \mathbb{F}_2)$ is monogenic. Then it is isomorphic to the cohomology algebra of a sphere or a projective space [11], [27; appendix 2, Theorems 6.2 and 6.3]. From the Leray–Serre spectral sequence of the path-loop fibration one easily computes the mod 2 cohomology of the loop space of M, showing in particular that the mod 2 Betti numbers are bounded. By the Leray–Serre spectral sequence of the base-point fibration the same follows for the mod 2 Betti numbers of the free loop space. But then the rational Betti numbers are also bounded since they are less than or equal to the mod 2 ones. Thus we see that none of the known sufficient conditions for infinitely many prime closed geodesics is satisfied. What do our results give? Let us look at the possible cases:

If M has the same mod 2 cohomology as the m-sphere then we just saw that 1.1 and 1.2 apply. In particular we have a map $f : S^m \to M$ which is non-trivial in $H_m(\,\cdot\,; \mathbb{F}_2)$. Hence in the present situation it even induces an isomorphism in all mod 2 homology groups, so that M actually has the 2-local homotopy type of the sphere [24; Chap. II, Theorem 1.14].

Otherwise M has the mod 2 cohomology of a projective space $\mathbb{F}P^m$. In order to apply Theorem 1.3 we would like to have a map $f : \mathbb{F}P^m \to M$ inducing an isomorphism in mod 2 (co)homology. The unique non-zero element in $H_d(M; \mathbb{F}_2)$ is the image of some $\alpha \in H_d(M; \mathbb{Z})$ (if not, by the universal coefficient theorem, $H_i(\,\cdot\,; \mathbb{F}_2)$ would be non-zero for $i = d - 1$ or $i = d + 1$). If, in addition, $\mathbb{F} = \mathbb{C}$ then α determines (up to homotopy) a map from M to $\mathbb{C}P^\infty$ since this is an Eilenberg-MacLane space of type $(\mathbb{Z}, 2)$. By cellular approximation we get a map $g : M \to \mathbb{C}P^m$ inducing an isomorphism in mod 2 (co)homology. Hence it is a 2-local homotopy equivalence. This does not mean that we always have a map in the opposite direction $f : \mathbb{C}P^m \to M$ inducing an isomorphism in mod 2 (co)homology. Thus, in general, 1.3 will not apply. If, however, the integral cohomology rings of M and $\mathbb{C}P^m$ are also isomorphic then g can be chosen to be a homotopy equivalence and f to be its inverse.

These considerations cannot be carried over to the cases $\mathbb{F} = \mathbb{H}$ and $\mathbb{F} = \mathrm{Ca}$. There are precisely 3 homotopy types of simply connected closed smooth manifolds with the same integer cohomology ring as the quaternionic projective plane (cf. [23; 2.1] or [18; Sect. 6] together with [19; Sect. 6]). If M is one of these manifolds and not homotopy equivalent to $\mathbb{H}P^2$ then, with the methods of the papers just mentioned, it is easy to see that there is not even a map $\mathbb{H}P^2 \to M$ inducing an isomorphism in mod 2 homology. For these manifolds M all the known methods of deriving lower bounds for the number of closed geodesics from the homotopy type are particularly ineffective: Since the integer cohomology ring is monogenic there is no chance for applying the Gromoll–Meyer Theorem. Although the cohomology is the same as for $\mathbb{H}P^2$ one cannot apply the methods developed for projective spaces ([25], [42] or our Theorem 1.3). What remains is only Theorem 1.1 with $m = 4$.

There should be analoguous examples with the cohomology of the Cayley plane or of quaternionic projective spaces in higher dimensions, but this seems not to be known [17; p. 413].

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